ECON 164: Theory of Economic Growth

Week 2: The Solow Model

Levi Crews

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Recap

- **three organizing questions**:
  1. Why are we so rich and they so poor?
  2. What is the engine of economic growth?
  3. How do "growth miracles" happen?

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  - growth rates are not necessarily constant
  - countries can move from "poor" to "rich" and vice versa
  - Kaldor facts (what characterizes "frontier" growth?) + Jones-Romer facts

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- **this week**: ***Solow model***
- **Week 3**: *Ramsey-Cass-Koopmans model*
- **Week 4**: *Neoclassical models* $\leftrightarrow$ *Facts*

What is a model?

A ***mathematical representation*** of some aspect of the economy. . .

- agents' objectives, constraints
- market structure, feasibility conditions
- parameters, exogenous variables

. . . that determines how we **interpret data** and **evaluate policy**

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So we need to make ***design choices***:

> *All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.* (Solow, 1956)

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(continuous time this week, discrete time next)

1. *How do economies produce?*

2. *How do economies accumulate factors of production?*

The Cobb-Douglas production function

$$\boxed{Y_t = K_t^{\alpha} \left(\_\_ L_t \right)^{1-\alpha}}$$

Y_t is **output**: "units of GDP"

- just *one* good

  (vs. many goods or multi-sector)
- no *international trade*

  (vs. trade, FDI, tech. diffusion)

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K_t is **capital**:

- in practice: durable physical inputs
  (machines, buildings, pencils, ...)
- in model: "units of GDP"

L_t is **labor**: headcount $\times$ hours

(later: human capital, skills, health ...)

Returns to scale

*What happens to output if we **double** K and L?*

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decreasing returns to scale	$F_t(\lambda K_t, \lambda L_t) < \lambda Y_t$	(fixed factor)
constant returns to scale	$F_t(\lambda K_t, \lambda L_t) = \lambda Y_t$	(replication)
increasing returns to scale	$F_t(\lambda K_t, \lambda L_t) > \lambda Y_t$	

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*What happens to output if we **double** K and L?*

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With Cobb-Douglas:

$$\begin{aligned} F_t(\lambda K_t, \lambda L_t) &= (\lambda K_t)^\alpha \left(\underline{\quad} \lambda L_t\right)^{1-\alpha} \\ &= \lambda^{\alpha+(1-\alpha)} (K_t)^\alpha \left(\underline{\quad} L_t\right)^{1-\alpha} \\ &= \boldsymbol{\lambda} Y_t \end{aligned}$$

Other characteristics of a *neoclassical* technology

2. **positive but diminishing returns**
3. **Inada conditions**
4. **essentiality**

(Barro and Sala-i Martin, 2004, pp. 26–28)

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$$\frac{\partial F_t}{\partial K_t} > 0, \qquad \frac{\partial^2 F_t}{\partial K_t^2} < 0$$

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$$\lim_{K_t \to 0} \frac{\partial F_t}{\partial K_t} = \lim_{L_t \to 0} \frac{\partial F_t}{\partial L_t} = \infty$$
$$\lim_{K_t \to \infty} \frac{\partial F_t}{\partial K_t} = \lim_{L_t \to \infty} \frac{\partial F_t}{\partial L_t} = 0$$

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Other characteristics of a *neoclassical* technology

2. **positive but diminishing returns**
3. **Inada conditions**
4. **essentiality** (implied by #1–3)

$$F_t(0, L_t) = F_t(K_t, 0) = 0$$

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Introducing "productivity"

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Specified as **labor-augmenting** ("Harrod neutral")—***does it matter?***

- in general, **yes**: (Barro and Sala-i Martin, 2004, pp.52–53, §1.5.3)
- with Cobb-Douglas, **no**: could just write $Y_t = \hat{A}_t K_t^\alpha L_t^{1-\alpha}$ w/ $\hat{A}_t = A_t^{1-\alpha}$

What about returns to scale? CRS in $\{K_t, L_t\}$, not $\{K_t, L_t, A_t\}$

The stage we've set

Growth can either be from...

- getting more K
- getting more L
- improving A

The representative firm

Assume a large number of identical firms
using the same Cobb-Douglas technology
maximize profits

$$\max_{K_t, L_t} \pi_t = Y_t - r_t K_t - w_t L_t,$$

taking factor prices as given:

- r_t is the rental rate of capital
- w_t is the wage rate

(why "large number of identical firms"?)

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Take **first-order conditions**:

$$\begin{aligned}
[K_t] \quad r_t &= \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} \\
&= \alpha \frac{Y_t}{K_t} \\
[L_t] \quad w_t &= (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} \\
&= (1-\alpha) \frac{Y_t}{L_t}
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\end{aligned}$$

so. . .

- **zero profits** (why? check!)
- **constant factor shares**:

$$\frac{r_t K_t}{Y_t} = \alpha \quad \text{and} \quad \frac{w_t L_t}{Y_t} = 1 - \alpha$$

Checklist: Kaldor facts

- ☐ output per worker ($y = Y/L$) should grow at a constant rate (g_y)
- ☐ capital per worker ($k = K/L$) should grow at a constant rate (g_k)
- ☐ the capital-output ratio (K/Y) should be constant
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Start with $Y_t = F_t(K_t, L_t)$, then totally differentiate and $\div$ by Y_t:

$$\begin{aligned}
\frac{dY_t}{Y_t} = \frac{\partial F_t}{\partial K_t}\frac{dK_t}{Y_t} + \frac{\partial F_t}{\partial L_t}\frac{dL_t}{Y_t} &= \frac{\partial F_t}{\partial K_t}\frac{K_t}{Y_t}\frac{dK_t}{K_t} + \frac{\partial F_t}{\partial L_t}\frac{L_t}{Y_t}\frac{dL_t}{L_t} \\
&= r_t\frac{K_t}{Y_t}\frac{dK_t}{K_t} + w_t\frac{L_t}{Y_t}\frac{dL_t}{L_t} \\
d\ln Y_t &= \frac{r_t K_t}{Y_t} d\ln K_t + \frac{w_t L_t}{Y_t} d\ln L_t \\
d\ln Y_t &\approx \alpha\, d\ln K_t + (1-\alpha)\, d\ln L_t \\
\boldsymbol{Y_t} &\boldsymbol{\approx K_t^{\alpha} L_t^{1-\alpha}}
\end{aligned}$$

(assumes perfect competition in factor markets)

Why *else* Cobb-Douglas?

*It's the result of **aggregating** many Leontief production functions** (Houthakker, 1955)

(*assumes Pareto distribution)

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A surprisingly **rich area for current research** (led by **UCLA**'s own **David Baqaee**):

- aggregation w/ growth (Jones, 2005)
- aggregation w/ any complementary inputs (Boehm and Oberfield, 2020)
- aggregation w/ distortions, nonlinearities (Baqaee and Farhi, 2019; Baqaee and Rubbo, 2023)

(*assumes Pareto distribution)

GDP per capita

For simplicity, let's assume *full labor force participation*. . . (cf. Table 1.1, Col. 3)

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$$\text{GDP per capita} = y_t \equiv \frac{Y_t}{L_t} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t$$

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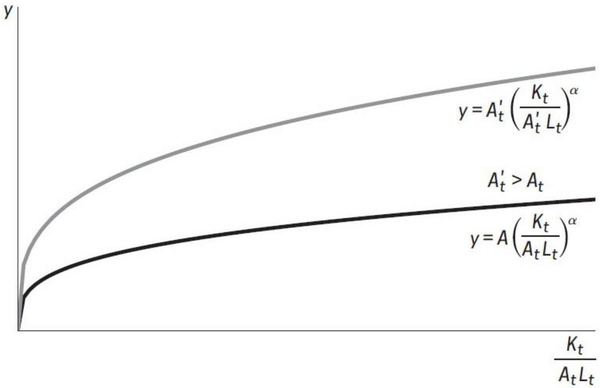
where we call $\boldsymbol{\tilde{k}_t}$ the stock of ***capital*** $(\boldsymbol{K_t})$ ***per efficiency unit of labor*** $(\boldsymbol{A_t L_t})$

GDP per capita

Figure 2.1 GDP per Capita as a Function of K_t/A_tL_t and Productivity



NOTE: The effects of a change in the K_t/A_tL_t ratio are captured by a movement along a given curve. An increase in productivity, as from A_t to A_t', is captured by a shift in the entire curve upward.

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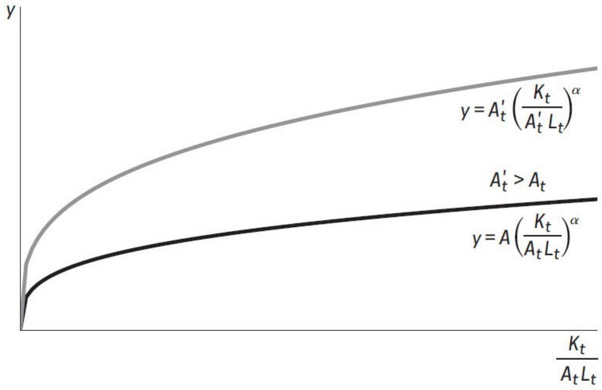

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(in eqbm: *indirect* effects matter, too!)

Growth rates: "take logs and derivatives"

For any variable x, let $\boldsymbol{g_x = \frac{d \ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

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Let's take **logs** then **derivatives** w.r.t. t of the GDP per capita equation:

$$\ln y_t = \alpha \ln \tilde{k}_t + \ln A_t$$

(right now, g_x could still be time-varying)

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\boldsymbol{g_y} &\boldsymbol{= \alpha(g_K - g_A - g_L) + g_A}
\end{aligned}$$

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The Solow model

1. ***How do economies produce?***

$$\boxed{Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}}$$

2. ***How do economies accumulate factors of production?***

Capital accumulation

$$\boxed{\dot{K}_t = I_t - \delta K_t}$$

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- **key assumption**: savings rate s is exogenous

(so $C_t = \ldots$?)

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- ☑ the share of expenditure going to capital (s) should be constant $\quad \boldsymbol{s \approx \frac{1}{4}}$
- ☐ wages (w) should grow at the same rate as output per worker

The growth rate of capital

Divide both sides of the capital accumulation equation by K_t:

$$\boldsymbol{g_K} = \frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} - \delta = s\frac{K_t^{\alpha}(A_tL_t)^{1-\alpha}}{K_t} - \delta$$

$$= s\left(\frac{A_tL_t}{K_t}\right)^{1-\alpha} - \delta$$

$$= \boldsymbol{s\tilde{k}_t^{\alpha-1} - \delta}$$

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How does $\boldsymbol{g_K}$ change with $\boldsymbol{\tilde{k}_t}$?

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How does $\boldsymbol{g_K}$ change with $\boldsymbol{\tilde{k}_t}$? $\downarrow$ b/c diminishing MPK ($\boldsymbol{\alpha < 1}$)

Closing the model

What about A_t and L_t?

Closing the model

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Assume exogenous growth rates:

$$A_t = A_0 \, e^{\boldsymbol{g_A} t}$$
$$L_t = L_0 \, e^{\boldsymbol{g_L} t}$$

(possibly zero!)

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$$L_t = L_0\, e^{\boldsymbol{g_L} t}$$

(possibly zero!)

Let's check...

$$\frac{\dot{\boldsymbol{A}}}{\boldsymbol{A}} = \frac{dA/dt}{A} = \frac{d\ln A_t}{dt} = \frac{d[\ln A_0 + g_A t]}{dt} = \boldsymbol{g_A}$$

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(A_t and L_t grow exogenously)

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Given. . .

- initial levels $\{K_0, A_0, L_0\}$
- parameters $\{\alpha, s, \delta, g_A, g_L\}$
- equations of the model

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. . . we obtain a **time path** for the **endogenous** variables $\{K_t, Y_t\}$ expressed in terms of the **exogenous** variables $\{A_t, L_t\}$, parameters, and initial conditions.

(full analytical solution in Jones and Vollrath (2024, Appx. B.2))

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Given...

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- parameters $\{\alpha, s, \delta, g_A, g_L\}$
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Check: Can you transform

$$g_K = s\tilde{k}_t^{\alpha-1} - \delta$$

into

$$\frac{d\tilde{k}_t}{dt} = s\tilde{k}_t^{\alpha} - (\delta + g_A + g_L)\tilde{k}_t$$

in two steps?

Focus on the path of $\tilde{k}_t$

Check: Can you transform

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into

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in two steps?

- ⋆ use $g_{\tilde{k}} = g_K - g_A - g_L$, then multiply by $\tilde{k}_t$

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in two steps?

Goal: find $\tilde{\boldsymbol{k}}^{\text{ss}}$ where $\boldsymbol{\frac{d\tilde{k}_t}{dt} = 0 = g_{\tilde{k}}}$
$\longrightarrow$ this is the ***steady state*** level

Focus on the path of $\tilde{k}_t$

Check: Can you transform

$$g_K = s\tilde{k}_t^{\alpha-1} - \delta$$

into

$$\frac{d\tilde{k}_t}{dt} = s\tilde{k}_t^{\alpha} - (\delta + g_A + g_L)\tilde{k}_t$$

in two steps?

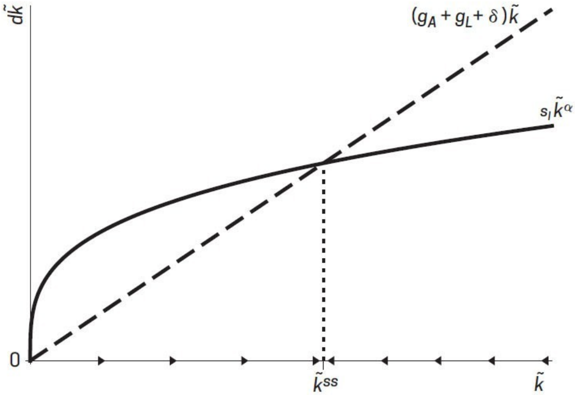
Goal: find $\tilde{\boldsymbol{k}}^{\text{ss}}$ where $\boldsymbol{\frac{d\tilde{k}_t}{dt} = 0 = g_{\tilde{k}}}$
$\longrightarrow$ this is the ***steady state*** level

What about the rest of the variables?

$$y_t = A_t\tilde{k}_t^{\alpha}$$
$$c_t = (1-s)y_t$$
$$w_t = (1-\alpha)y_t$$
$$r_t = \alpha\tilde{k}_t^{\alpha-1}$$
$$\vdots$$
$$\frac{K_t}{Y_t} = \tilde{k}_t^{1-\alpha}$$

The traditional phase diagram ($d\tilde{k}/dt$ vs. $\tilde{k}$)

Figure B.1 The Traditional Solow Diagram



NOTE: The figure shows the accumulation of $\tilde{k}$, through $s_I\tilde{k}^{\alpha}$ as the black line. The decrease in $\tilde{k}$, $(\delta + g_A + g_L)\tilde{k}$, is shown as the dashed line. From equation (B.1) we know that the absolute change in $\tilde{k}$ is the difference between the two lines, so that everywhere $s_I\tilde{k}^{\alpha} > (\delta + g_A + g_L)\tilde{k}$ (e.g., to the left of the intersection) $d\tilde{k} > 0$, and everywhere that $s_I\tilde{k}^{\alpha} < (\delta + g_A + g_L)\tilde{k}$ (e.g., to the right of the intersection) $d\tilde{k} < 0$. There is a steady state at the intersection where $d\tilde{k} = 0$.

Solve for $\tilde{k}^{ss}$:

$$\frac{d\tilde{k}_t}{dt} = s\tilde{k}_t^{\alpha} - (\delta + g_A + g_L)\tilde{k}_t$$

The traditional phase diagram ($d\tilde{k}/dt$ vs. $\tilde{k}$)

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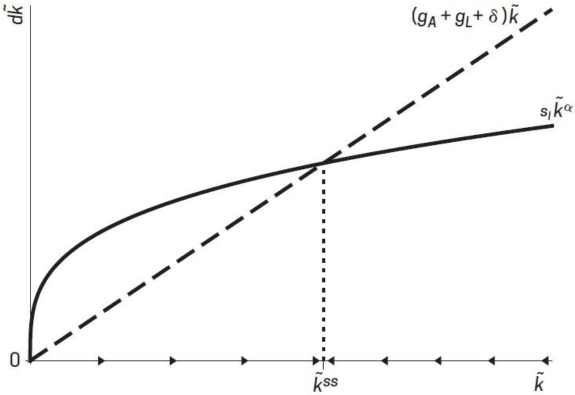

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Solve for $\tilde{k}^{ss}$:

$$\frac{d\tilde{k}_t}{dt} = s\tilde{k}_t^{\alpha} - (\delta + g_A + g_L)\tilde{k}_t$$

$$0 = s\left(\tilde{k}^{ss}\right)^{\alpha} - (\delta + g_A + g_L)\tilde{k}^{ss}$$

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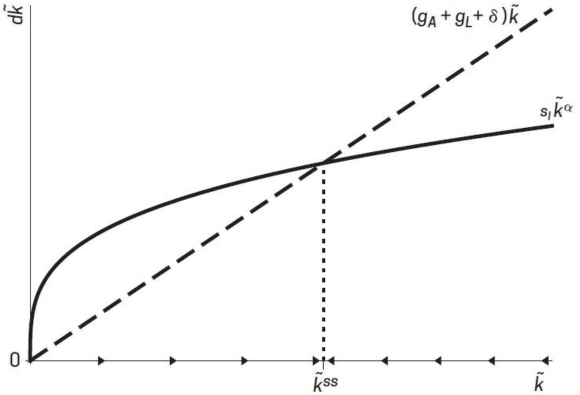

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$$\tilde{k}^{ss} = \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}}$$

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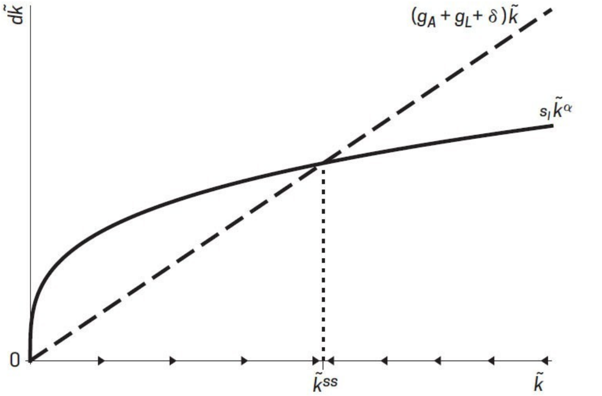
Figure B.1 The Traditional Solow Diagram



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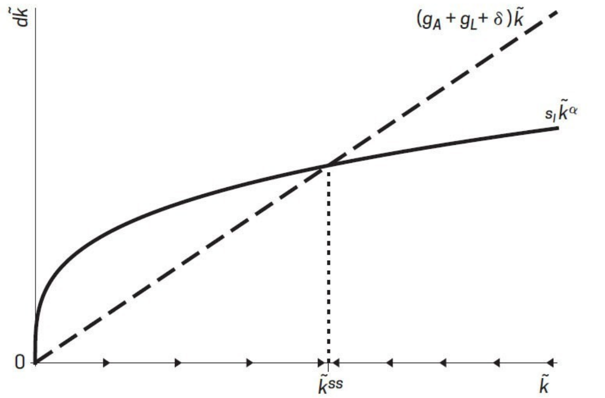
$$\tilde{k}^{ss} = \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}}$$

What happens if $\tilde{k}_0 < \tilde{k}^{ss}$?

Figure B.1 The Traditional Solow Diagram



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$$0 = s\left(\tilde{k}^{ss}\right)^{\alpha} - (\delta + g_A + g_L)\tilde{k}^{ss}$$

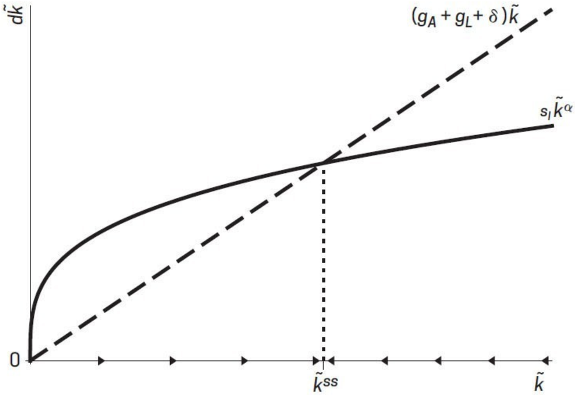
$$\tilde{k}^{ss} = \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}}$$

What happens if $\tilde{k}_0 < \tilde{k}^{ss}$? $\boldsymbol{\tilde{k}_t} \nearrow$

What happens if $\tilde{k}_0 > \tilde{k}^{ss}$?

The traditional phase diagram ($d\tilde{k}/dt$ vs. $\tilde{k}$)

Figure B.1 The Traditional Solow Diagram



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$$0 = s\left(\tilde{k}^{ss}\right)^{\alpha} - (\delta + g_A + g_L)\tilde{k}^{ss}$$

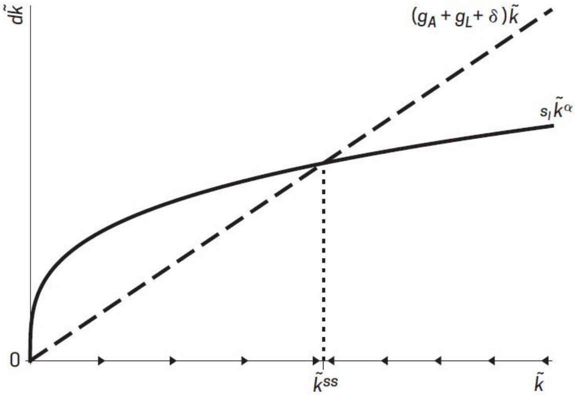
$$\tilde{k}^{ss} = \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}}$$

What happens if $\tilde{k}_0 < \tilde{k}^{ss}$? $\boldsymbol{\tilde{k}_t} \nearrow$

What happens if $\tilde{k}_0 > \tilde{k}^{ss}$? $\boldsymbol{\tilde{k}_t} \swarrow$

The traditional phase diagram ($d\tilde{k}/dt$ vs. $\tilde{k}$)

Figure B.1 The Traditional Solow Diagram



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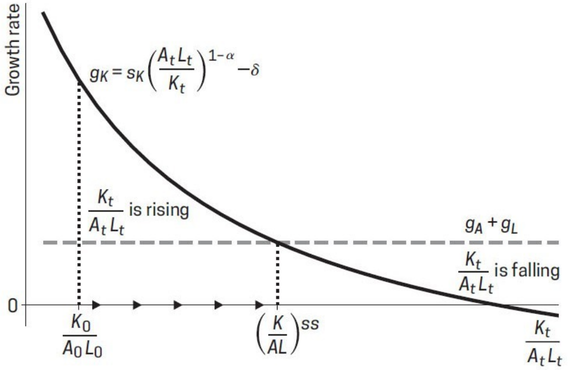
What happens if $\tilde{k}_0 < \tilde{k}^{ss}$? $\boldsymbol{\tilde{k}_t} \nearrow$

What happens if $\tilde{k}_0 > \tilde{k}^{ss}$? $\boldsymbol{\tilde{k}_t} \swarrow$

How quickly?

The Jones-Vollrath phase diagram (g_K vs. $\tilde{k}$)

Figure 2.2 The Dynamics of the K_t/A_tL_t Ratio



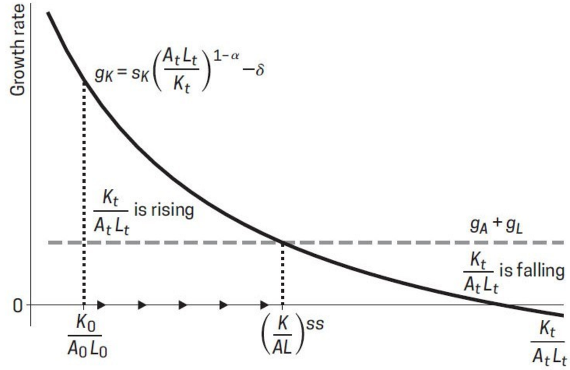
NOTE: The dark line plots the growth rate of capital, g_K, from equation (2.11) against the K_t/A_tL_t ratio. The growth rate of AL, which is $g_A + g_L$, is plotted in the dashed line. Everywhere the dashed line is above the solid line, the K_t/A_tL_t ratio must be falling, as $g_K < g_A + g_L$. Everywhere the dashed line is below the solid line, the K_t/A_tL_t ratio must be rising, as $g_K > g_A + g_L$.

Solve for $\tilde{k}^{ss}$:

$$g_{\tilde{k}} = g_K - g_A - g_L$$

The Jones-Vollrath phase diagram (g_K vs. $\tilde{k}$)

Figure 2.2 The Dynamics of the K_t/A_tL_t Ratio



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Solve for $\tilde{k}^{ss}$:

$$g_{\tilde{k}} = g_K - g_A - g_L$$
$$0 = s\left(\tilde{k}^{ss}\right)^{\alpha-1} - \delta - g_A - g_L$$

The Jones-Vollrath phase diagram (g_K vs. $\tilde{k}$)

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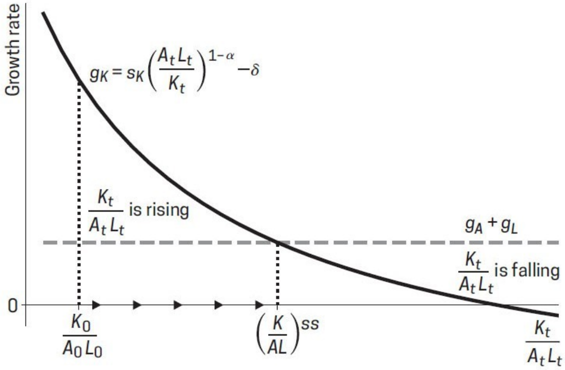

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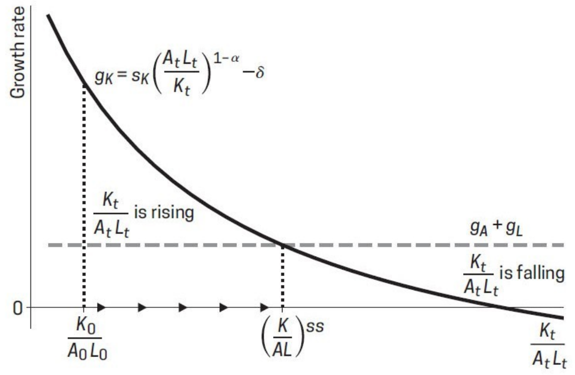
The Jones-Vollrath phase diagram (g_K vs. $\tilde{k}$)



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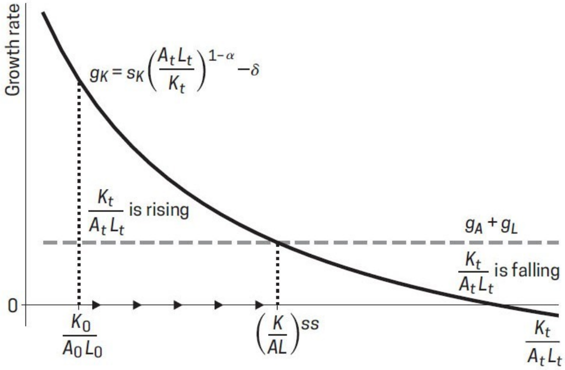
Solve for $\tilde{k}^{ss}$:

$$
\begin{aligned}
g_{\tilde{k}} &= g_K - g_A - g_L \\
0 &= s\left(\tilde{k}^{ss}\right)^{\alpha-1} - \delta - g_A - g_L \\
\tilde{k}^{ss} &= \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Same transitional dynamics . . .

Figure 2.2 The Dynamics of the K_t/A_tL_t Ratio



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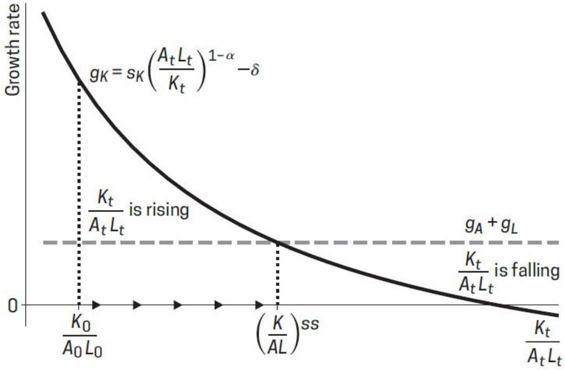
Same transitional dynamics . . .

How quickly?

The Jones-Vollrath phase diagram (g_K vs. $\tilde{k}$)

Figure 2.2 The Dynamics of the K_t/A_tL_t Ratio



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$$\tilde{k}^{ss} = \left(\frac{s}{\delta + g_A + g_L}\right)^{\frac{1}{1-\alpha}}$$

Same transitional dynamics . . .

How quickly? Read it off the gap!

(bigger gap = faster transition)

The balanced growth path (BGP)

Remember what we care about: ***GDP per capita***

$$g_y^{\mathsf{ss}} = \alpha(g_K^{\mathsf{ss}} - g_A - g_L) + g_A$$

The balanced growth path (BGP)

Remember what we care about: ***GDP per capita***

$$g_y^{\mathsf{ss}} = \alpha(\underbrace{g_K^{\mathsf{ss}} - g_A - g_L}_{=0}) + g_A$$

The balanced growth path (BGP)

Remember what we care about: *GDP per capita*

$$g_y^{\text{ss}} = \boldsymbol{g_A}$$

The balanced growth path (BGP)

Remember what we care about: ***GDP per capita***

$$g_y^{\text{ss}} = \boldsymbol{g_A}$$

What about aggregates? $Y_t = y_t L_t$, $C_t = (1-s)Y_t$, $K_t = \tilde{k}_t(A_t L_t)$

$$g_Y^{\text{ss}} = g_C^{\text{ss}} = g_K^{\text{ss}} = \boldsymbol{g_A + g_L}$$

The balanced growth path (BGP)

Remember what we care about: ***GDP per capita***

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$$g_Y^{\textsf{ss}} = g_C^{\textsf{ss}} = g_K^{\textsf{ss}} = \boldsymbol{g_A + g_L}$$

All steady-state growth here is exogenous!

Checklist: Kaldor facts

- ☐ output per worker ($y = Y/L$) should grow at a constant rate
- ☐ capital per worker ($k = K/L$) should grow at a constant rate
- ☐ the capital-output ratio (K/Y) should be constant
- ☐ the rate of return on capital (r) should be constant
- ☑ the share of income going to labor (wL/Y) should be constant $\quad \boldsymbol{1-\alpha \approx \frac{2}{3}}$
- ☑ the share of expenditure going to capital (s) should be constant $\quad \boldsymbol{s \approx \frac{1}{4}}$
- ☐ wages (w) should grow at the same rate as output per worker

Checklist: Kaldor facts

- ☑ output per worker ($y = Y/L$) should grow at a constant rate $\quad g_y^{\text{ss}} = g_A$
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Checklist: Kaldor facts

- ☑ output per worker ($y = Y/L$) should grow at a constant rate $\quad g_y^{\text{ss}} = g_A$
- ☑ capital per worker ($k = K/L$) should grow at a constant rate $\quad g_k^{\text{ss}} = g_{\tilde{k}}^{\text{ss}} + g_A$
- ☐ the capital-output ratio (K/Y) should be constant
- ☐ the rate of return on capital (r) should be constant
- ☑ the share of income going to labor (wL/Y) should be constant $\quad 1-\alpha \approx \frac{2}{3}$
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Checklist: Kaldor facts

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- ☑ the share of income going to labor (wL/Y) should be constant $\boldsymbol{1-\alpha \approx \frac{2}{3}}$
- ☑ the share of expenditure going to capital (s) should be constant $\boldsymbol{s \approx \frac{1}{4}}$
- ☐ wages (w) should grow at the same rate as output per worker

Checklist: Kaldor facts

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- ☑ capital per worker ($k = K/L$) should grow at a constant rate $\quad g_k^{ss} = g_A$
- ☑ the capital-output ratio (K/Y) should be constant $\quad K^{ss}/Y^{ss} = (\tilde{k}^{ss})^{1-\alpha}$
- ☑ the rate of return on capital (r) should be constant $\quad r^{ss} = \alpha(\tilde{k}^{ss})^{\alpha-1}$
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- ☑ the share of expenditure going to capital (s) should be constant $\quad s \approx \frac{1}{4}$
- ☑ wages (w) should grow at the same rate as output per worker $\quad g_w^{ss} = g_y^{ss}$

Comparative statics → transitional dynamics

What happens if we change s?

What happens if we change g_L?

What happens if we destroy some K?

Comparative statics $\rightarrow$ transitional dynamics

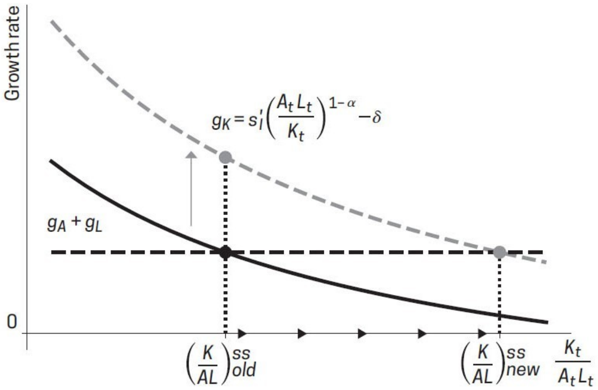
What happens if we change s?

What happens if we change g_L?

What happens if we destroy some K? ***Problem Set #2***

An increase in s: What happens to $\tilde{k}_t$?

Figure 2.3 The Dynamics of an Increase in s_I

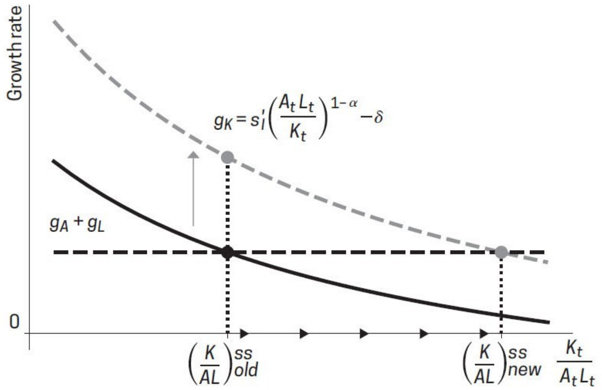


NOTE: The dark line plots the growth rate of capital, g_K, from equation (2.11) against the K_t/A_tL_t ratio for the old situation. The gray line plots g_K when the capital formation rate increases to s_I'. The growth rate of productivity and labor, $g_A + g_L$, does not change. In response to the increase in the capital formation rate, the ratio K_t/A_tL_t moves over time from the old steady state to the new, higher, steady state.

Say s jumps to s' at $t = T^*$...

An increase in s: What happens to $\tilde{k}_t$?

Figure 2.3 The Dynamics of an Increase in s_I



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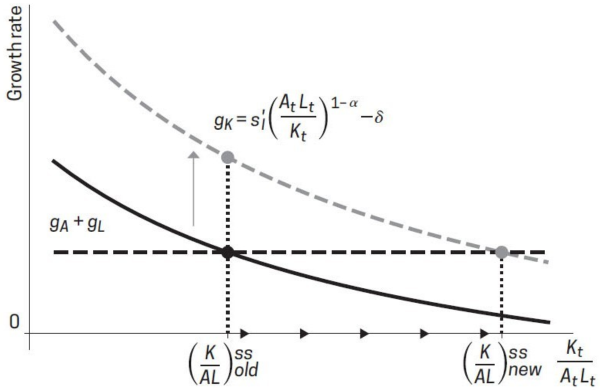
Say s jumps to s' at $t = T^*$...

1. What is $\tilde{k}_{T^*}$?

An increase in s: What happens to $\tilde{k}_t$?

Figure 2.3 The Dynamics of an Increase in s_I



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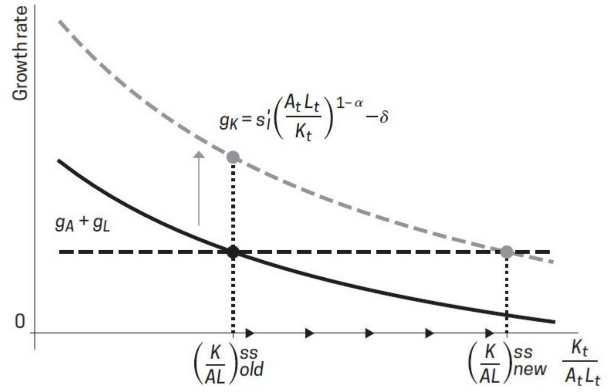
Say s jumps to s' at $t = T^*$...

1. What is $\tilde{k}_{T^*}$?
2. What is the new $\tilde{k}^{ss}$?

An increase in s: What happens to $\tilde{k}_t$?

Figure 2.3 The Dynamics of an Increase in s_I



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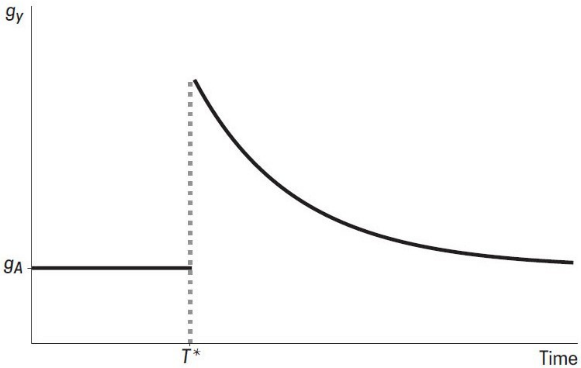
Say s jumps to s' at $t = T^*$...

1. What is $\tilde{k}_{T^*}$?
2. What is the new $\tilde{k}^{\text{ss}}$?
3. How do we get there?
   - does $\tilde{k}_t$ jump immediately?
   - when is growth fastest?

Figure 2.4 The Growth Rate of GDP per Capita after Change in s_I

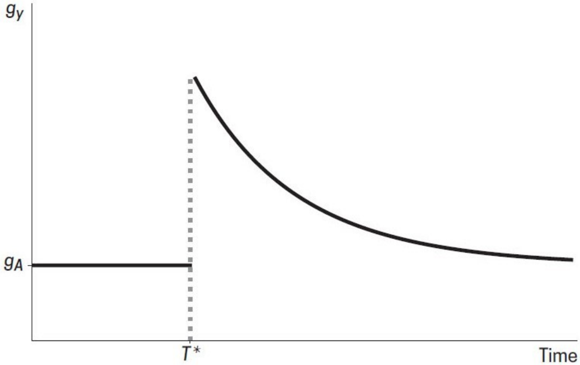


NOTE: In the plot, the shift to s_I' happens at time T^*. At that point the growth rate of GDP per capita jumps, but that increase is only temporary as the economy approaches the new steady state. Eventually the growth rate returns to equal g_A.

$$g_y = \alpha(g_K - g_A - g_L) + g_A$$

An increase in s: What happens to g_y?

Figure 2.4 The Growth Rate of GDP per Capita after Change in s_I



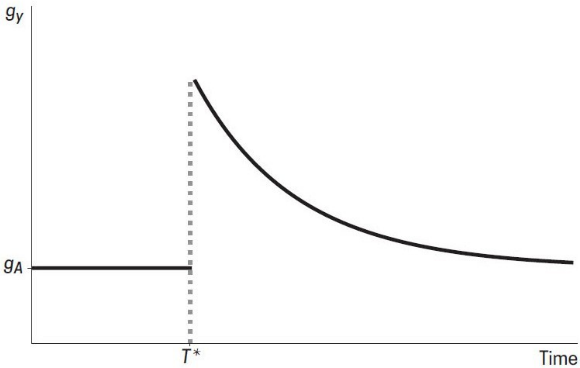
NOTE: In the plot, the shift to s_I' happens at time T^*. At that point the growth rate of GDP per capita jumps, but that increase is only temporary as the economy approaches the new steady state. Eventually the growth rate returns to equal g_A.

$$g_y = \alpha(g_K - g_A - g_L) + g_A$$

- at T^*: $\quad g_K > g_A + g_L$

An increase in s: What happens to g_y?

Figure 2.4 The Growth Rate of GDP per Capita after Change in s_I



NOTE: In the plot, the shift to s_I' happens at time T^*. At that point the growth rate of GDP per capita jumps, but that increase is only temporary as the economy approaches the new steady state. Eventually the growth rate returns to equal g_A.

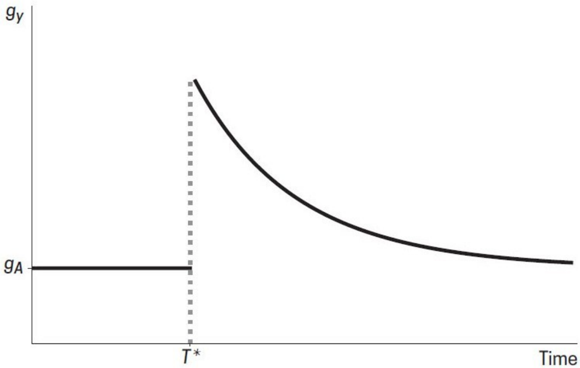
$$g_y = \alpha(g_K - g_A - g_L) + g_A$$

- at T^*: $g_K > g_A + g_L$
- as $t \to \infty$: $g_K \searrow g_A + g_L$

(why? diminishing MPK)

An increase in s: What happens to g_y?

Figure 2.4 The Growth Rate of GDP per Capita after Change in s_I



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$$g_y = \alpha(g_K - g_A - g_L) + g_A$$

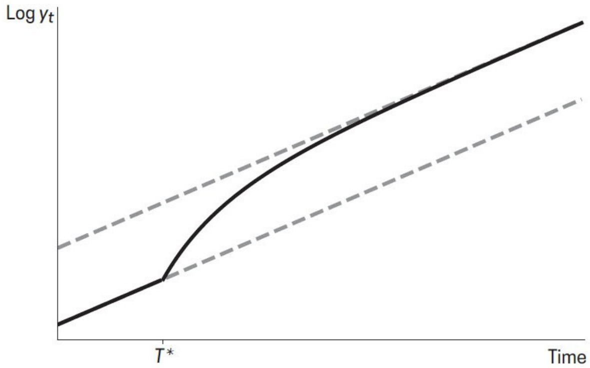
- at T^*: $g_K > g_A + g_L$
- as $t \to \infty$: $g_K \searrow g_A + g_L$

(why? diminishing MPK)

Faster growth is temporary!

An increase in s: What happens to y?

Figure 2.5 The Level of GDP per Capita after Change in s_I



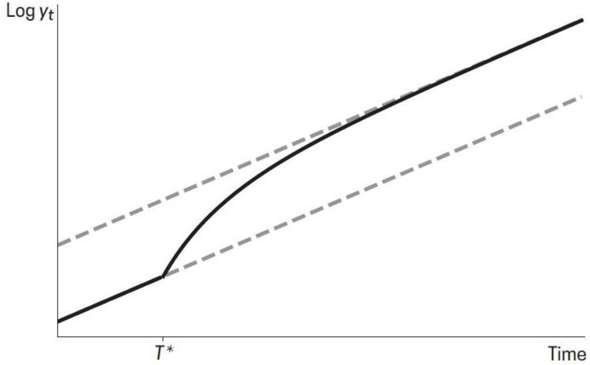
NOTE: The y-axis measures the log of GDP per capita, and the x-axis measures time. So the slope of the line indicates the growth rate of GDP per capita. The dark line plots the actual path of GDP per capita in response to the shift to s_I' at time T^*. The gray dashed lines indicate the hypothetical balanced growth paths at s_I (the lower one) and s_I' (the higher one).

On a **balanced growth path** (---):

$$y_t^{\text{BGP}} = A_t \left(\tilde{k}^{\text{ss}}\right)^{\alpha}$$

An increase in s: What happens to y?

Figure 2.5 The Level of GDP per Capita after Change in s_I



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On a **balanced growth path** (---):

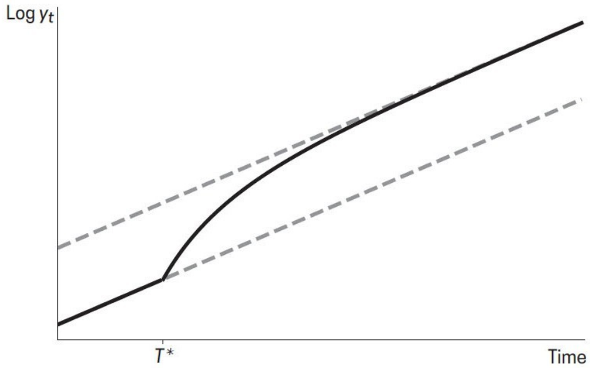
$$y_t^{\text{BGP}} = A_t \left(\tilde{k}^{\text{ss}}\right)^{\alpha}$$

$$\ln y_t^{\text{BGP}} = g_A\, t + \ln A_0$$

$$+ \frac{\alpha}{1-\alpha} \ln \left[\frac{s}{\delta+g_A+g_L}\right]$$

An increase in s: What happens to y?

Figure 2.5 The Level of GDP per Capita after Change in s_l



NOTE: The y-axis measures the log of GDP per capita, and the x-axis measures time. So the slope of the line indicates the growth rate of GDP per capita. The dark line plots the actual path of GDP per capita in response to the shift to s_l' at time T^*. The gray dashed lines indicate the hypothetical balanced growth paths at s_l (the lower one) and s_l' (the higher one).

On a **balanced growth path** (---):

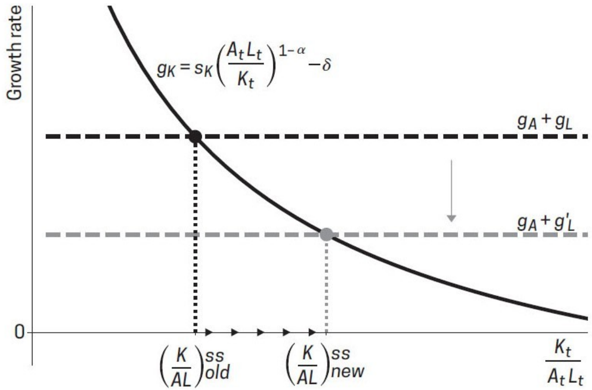
$$y_t^{\text{BGP}} = A_t \left(\tilde{k}^{\text{ss}}\right)^{\alpha}$$

$$\ln y_t^{\text{BGP}} = g_A\, t + \ln A_0 + \frac{\alpha}{1-\alpha} \ln \left[\frac{s'}{\delta + g_A + g_L}\right]$$

Higher level is permanent!

A decline in g_L: What happens to $\tilde{k}_t$?

Figure 2.6 The Dynamics of a Decrease in g_L



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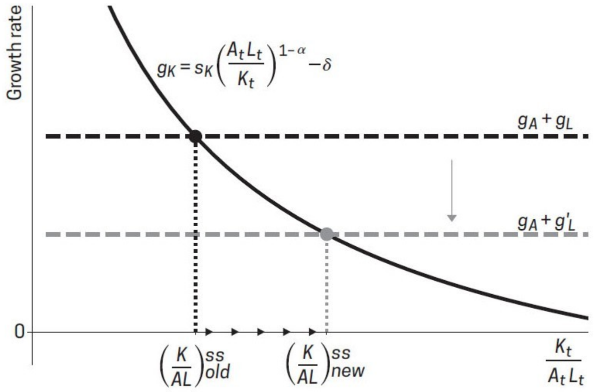
Say g_L falls to g'_L at $t = T^* \ldots$

A decline in g_L: What happens to $\tilde{k}_t$?

Figure 2.6 The Dynamics of a Decrease in g_L



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Say g_L falls to g'_L at $t = T^*$...

1. What is $\tilde{k}_{T^*}$?

A decline in g_L: What happens to $\tilde{k}_t$?

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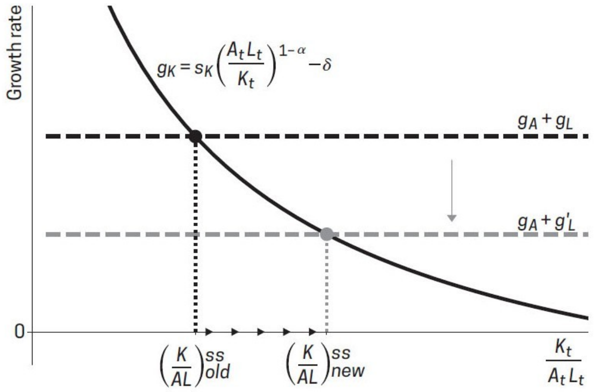

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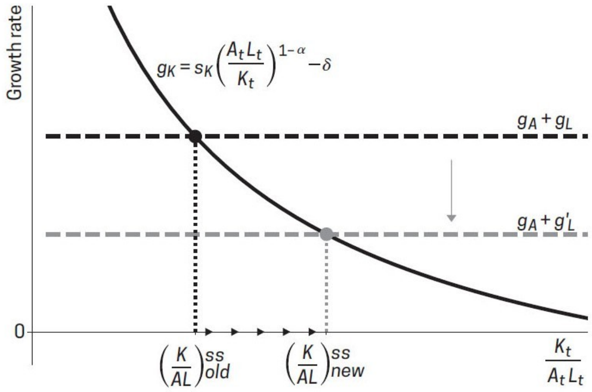

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Say g_L falls to g'_L at $t = T^*$...

1. What is $\tilde{k}_{T^*}$?
2. What is the new $\tilde{k}^{ss}$?
3. How do we get there?
   - does $\tilde{k}_t$ jump immediately?
   - when is growth fastest?

A decline in g_L: What happens to $\tilde{k}_t$?

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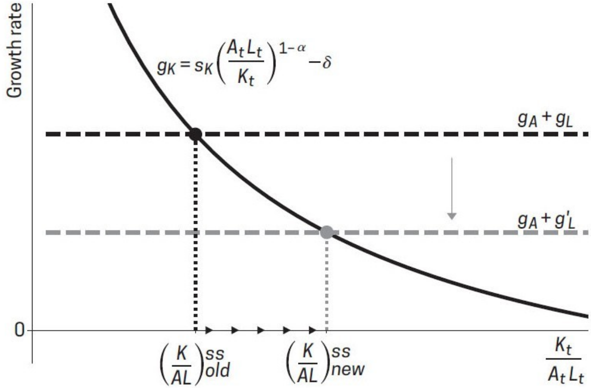

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What happens to g_y and y?

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