

# ECON 164: Theory of Economic Growth

## Week 3: The Neoclassical Growth Model

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# Recap

- Last time, we developed the **Solow model**, which has two building blocks:
  - the aggregate production function:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

- **endogenous** capital accumulation...

$$\dot{K}_t = sY_t - \delta K_t$$

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- But this picture is incomplete: *what determines variation in  $s$ ?*
- This week: Solow w/ **endogenous** saving → Ramsey-Cass-Koopmans model

(in discrete time, h/t *Adrien Bilal*)

## What are we saving *for*?

- Endogenizing saving behavior requires taking a stance on what we're saving for
- Most of us save to **spend** more later!
- This is why we'll be analyzing **consumption** and **saving *jointly***

## What should go into a model of consumption/saving?

- Solow model: consumption/saving only depend on **current income**  $Y_t$

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- Take *microfounded* approach: saving is **chosen** by a rational utility-maximizing rep. HH
- **This week:** in *discrete* time...
  - two periods (ECON 102)
  - $\infty$  periods
  - ***back to growth!***

## Two-period consumption-saving

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Consider the behavior of a **representative household** over two periods ( $t = 1, 2$ )

# Utility function

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  - assume *positive* but *decreasing* ( $\rightarrow U$  is strictly concave) [interpretation?]
  - e.g., if  $U(C) = \ln C$  then  $U'(C) = \frac{1}{C}$

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  - assume *positive* but *decreasing* ( $\rightarrow U$  is strictly concave) [interpretation?]
  - e.g., if  $U(C) = \ln C$  then  $U'(C) = \frac{1}{C}$
- $U(C_1) + \beta U(C_2) \equiv$  **lifetime utility**, where  $\beta \in (0, 1)$  is the discount factor

# Budget constraint

Suppose the household receives income  $Y_t$  at date  $t$  for  $t = 1, 2$

- saving  $S$  earns interest rate  $r$
- the **per-period (flow)** budget constraints:

$$C_1 + S = Y_1$$

$$C_2 = Y_2 + (1 + r)S$$

- How to turn these *two* constraints into a single one?



## Present discounted value

The present discounted value (PDV) of a payment  $D$  at date  $t$ ...

*how much you need to save at date 0 to have  $D$  at date  $t$*

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- Examples:

- To have \$1 at  $t = 1$ , you need to save  $\text{PDV} = \frac{1}{1+r}$  at  $t = 0$
- To have \$1 at  $t = 2$ , you need to save  $\text{PDV} = \frac{1}{(1+r)^2}$  at  $t = 0$

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  - To have \$1 at  $t = 2$ , you need to save  $\text{PDV} = \frac{1}{(1+r)^2}$  at  $t = 0$
- What if you get \$1 each year from this year onward?

$$\text{PDV} = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots = \underbrace{\sum_{i=0}^{\infty} \left( \frac{1}{1+r} \right)^i}_{\text{geometric sum}} = \frac{1}{1 - \frac{1}{1+r}} = \frac{1+r}{r}$$

# The lifetime budget constraint

Combine the per-period budget constraints into a single **lifetime budget constraint**:

1. Divide the second period constraint by  $(1 + r)$ :

$$C_2 = Y_2 + (1 + r)S \quad \rightarrow \quad \frac{C_2}{1 + r} = \frac{Y_2}{1 + r} + S$$

2. Substitute in  $S = Y_1 - C_1$ :

$$\underbrace{C_1 + \frac{C_2}{1 + r}}_{\text{PDV of consumption}} = \underbrace{Y_1 + \frac{Y_2}{1 + r}}_{\text{PDV of income}}$$

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Coming up: *this holds more generally!*

## Two formulations of the two-period consumption-saving problem

1. The **per-period** formulation:

$$\begin{aligned} \max_{C_1, C_2, S} \quad & U(C_1) + \beta U(C_2) \quad \text{s.t.} \quad C_1 + S = Y_1 \\ & C_2 = Y_2 + (1 + r)S \end{aligned}$$

2. The **lifetime** formulation:

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## Two ways to solve

**Substitution**  $\rightarrow$  *unconstrained* optimization  
using per-period budget constraints. . .

$$\max_S U(Y_1 - S) + \beta U(Y_2 + (1 + r)S)$$

$$\rightarrow U'(Y_1 - S) = \beta(1 + r)U'(Y_2 + (1 + r)S)$$

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**Lagrangian** = constrained optimization  
using lifetime budget constraint. . .

$$\begin{aligned} \max_{C_1, C_2} \quad & U(C_1) + \beta U(C_2) \\ & + \lambda \left( Y_1 + \frac{Y_2}{1+r} - C_1 - \frac{C_2}{1+r} \right) \\ \rightarrow \quad & U'(C_1) = \lambda, \quad \beta U'(C_2) = \frac{\lambda}{1+r} \end{aligned}$$



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Either way, end up with the same **Euler equation**:

$$\underbrace{U'(C_1)}_{\text{MU if consume one more unit this period}} = \underbrace{\beta(1+r)U'(C_2)}_{\substack{\text{MU if save one more unit this period} \\ \rightarrow \text{consume } 1+r \text{ more next period}}}$$

With the budget constraint(s), uniquely pins down  $C_1, C_2, S$

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so period-1 savings are

$$S = Y_1 - C_1 = \frac{Y_1 - Y_2}{2 + r}$$

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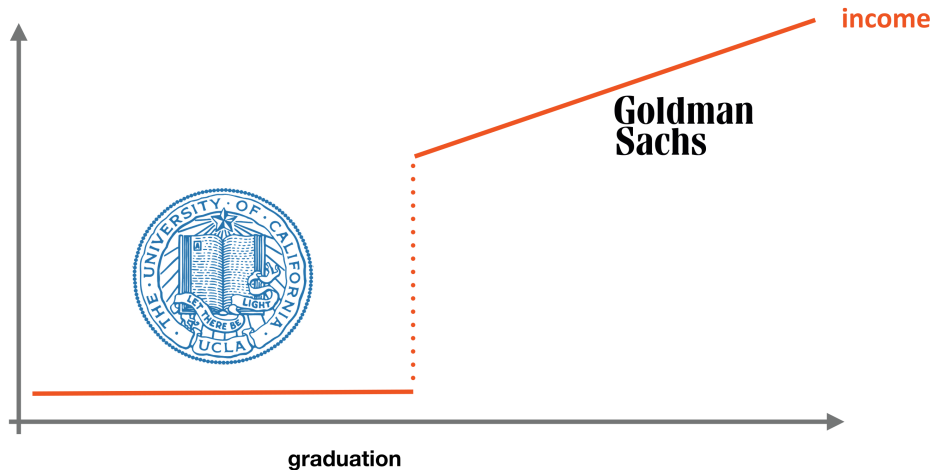
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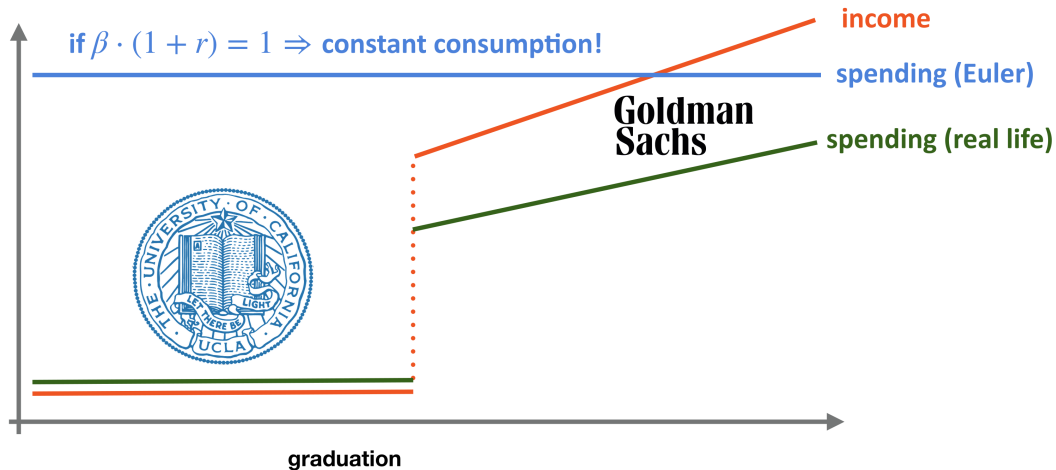
so period-1 savings are

$$S = Y_1 - C_1 = \frac{\beta Y_1 - \frac{Y_2}{1+r}}{1+\beta}$$

# Are *you* on your Euler equation?



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## Infinite-horizon consumption-saving

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# Infinite-horizon utility function

- **Why  $\infty$ ?** Clearly people do not live forever...
  - Yes, but each generation does care about the next one  $\rightarrow$  **saves to leave bequests**
  - So could interpret each person as part of an “infinite dynasty”
  - Practically, the math for infinite lives is nicer than for finite lives
- Label periods by  $t = 0, 1, 2, \dots$  (start at date 0). Consumption at date  $t \equiv C_t$ .
- So now **lifetime utility** is

$$U(C_0) + \beta U(C_1) + \beta^2 U(C_2) + \dots = \sum_{t=0}^{\infty} \beta^t U(C_t)$$

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(can we do this if  $\beta = 1$ ?)

# Infinite-horizon per-period budget constraints

- Now agents can save over **many periods**
  - $S_t \equiv$  stock of savings going into period  $t$ , earns interest rate  $r$
  - $S_0 \equiv$  *initial wealth* at the beginning of period 0
- Get an **infinite sequence of per-period (flow) budget constraints**

$$C_0 + S_1 = (1 + r)S_0 + Y_0$$

$$C_1 + S_2 = (1 + r)S_1 + Y_1$$

$$C_2 + S_3 = (1 + r)S_2 + Y_2$$

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$$C_t + S_{t+1} = (1 + r)S_t + Y_t \quad \text{for } t = 0, 1, 2, \dots$$

## Infinite-horizon lifetime budget constraint

- We can again combine all the per-period constraints into a **lifetime constraint**
- Solve out per-period consumption  $C_t$  and take **PDVs** from date 0

$$C_0 = (1+r)S_0 + Y_0 - S_1 \quad \rightarrow \quad C_0 = (1+r)S_0 + Y_0 - S_1$$

$$C_1 = (1+r)S_1 + Y_1 - S_2 \quad \rightarrow \quad \frac{C_1}{1+r} = S_1 + \frac{Y_1}{1+r} - \frac{S_2}{1+r}$$

$$C_2 = (1+r)S_2 + Y_2 - S_3 \quad \rightarrow \quad \frac{C_2}{(1+r)^2} = \frac{S_2}{1+r} + \frac{Y_2}{(1+r)^2} - \frac{S_3}{(1+r)^2}$$

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$$\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}$$



# Two formulations of the infinite-horizon consumption-saving problem

1. The **per-period** formulation:

$$\max_{\{C_t, S_t\}} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.} \quad C_t + S_{t+1} = (1+r)S_t + Y_t \quad \text{for } t = 0, 1, 2, \dots$$

2. The **lifetime** formulation:

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2. The **lifetime** formulation:

solve via **Lagrangian**

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}$$

*continuous time: solve via **Hamiltonian***

(Barro and Sala-i Martin, 2004)

The Lagrangian for the lifetime formulation is

$$\mathcal{L}(\{C_t\}) = \sum_{t=0}^{\infty} \beta^t U(C_t) + \lambda \left[ (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} - \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} \right]$$

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The first-order conditions w.r.t. any particular  $C_t$  and  $C_{t+1}$  are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial C_t} = \beta^t U'(C_t) - \lambda \frac{1}{(1+r)^t} \\ 0 &= \frac{\partial \mathcal{L}}{\partial C_{t+1}} = \beta^{t+1} U'(C_{t+1}) - \lambda \frac{1}{(1+r)^{t+1}} \end{aligned}$$

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Solving for  $\lambda$  and rearranging recovers the **Euler equation**: for  $t = 0, 1, 2, \dots$

$$U'(C_t) = \beta(1+r)U'(C_{t+1})$$

## Characterizing the solution

The **solution** is then the unique set of  $C_0, C_1, C_2, \dots$  that solve

$$U'(C_t) = \beta(1+r)U'(C_{t+1}) \quad \forall t, \quad \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}$$

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With **log utility**, the Euler equation simplifies to  $C_{t+1} = \beta(1+r)C_t$ , so

$$C_t = \beta^t(1+r)^t C_0$$

$$\begin{aligned} C_0 &= \left( (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} \right) / \left( \sum_{t=0}^{\infty} \beta^t \right) \\ &= \left( (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} \right) / \left( \frac{1}{1-\beta} \right) \\ &= (1-\beta) \left( (1+r)S_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t} \right) \end{aligned}$$

- What *e*/se should  $C_t$  and  $I_t$  depend on other than **current income  $Y_t$** ?



# Taking stock

- What *else* should  $C_t$  and  $I_t$  depend on other than **current income**  $Y_t$ ?
- Take *microfounded* approach:  
saving is **chosen** by a rational utility-maximizing rep. HH over  $\infty$  periods

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So, turning back to **growth**...

**opportunities** vs. **necessities**

(modern macro inc. all these *and more!*)

## Revisiting the Solow model

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# Consumption in the Solow model, revisited

Recall from last week...

$$y_t^{\text{BGP}} = A_t \left( \tilde{k}^{\text{ss}} \right)^\alpha = A_t \left( \frac{s}{\delta + g_A + g_L} \right)^{\frac{\alpha}{1-\alpha}}$$

...so should we just max out w/  $s = 1$ ?

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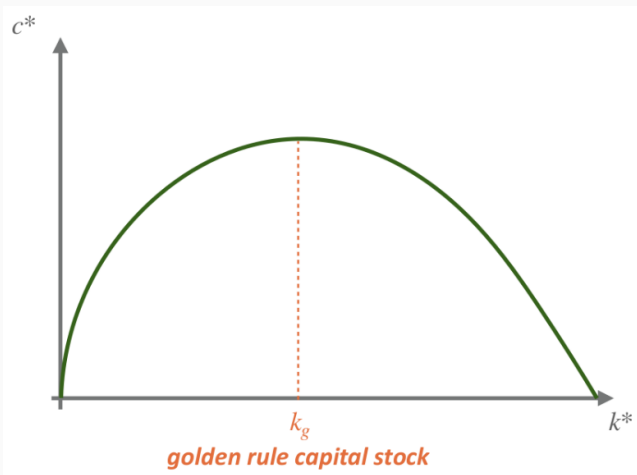
...so should we just max out w/  $s = 1$ ?

Not if what matters for utility is **consumption**:

$$c_t^{\text{BGP}} = (1 - s)y_t^{\text{BGP}} = (1 - s)A_t \left( \frac{s}{\delta + g_A + g_L} \right)^{\frac{\alpha}{1-\alpha}}$$

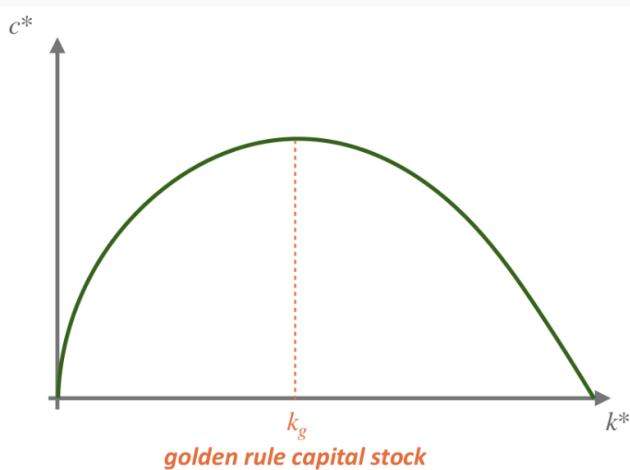
# The “golden rule” to optimize steady-state consumption

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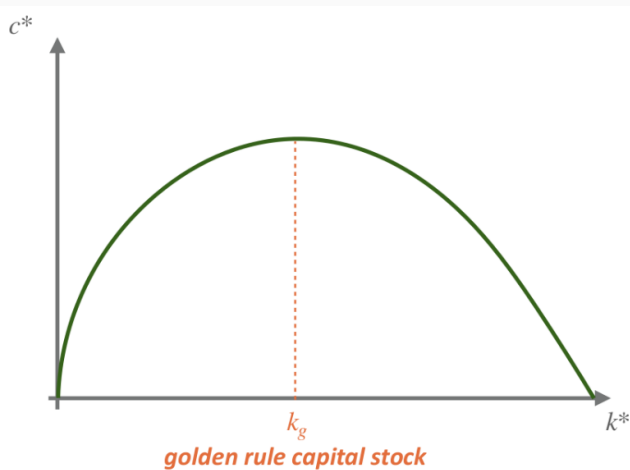
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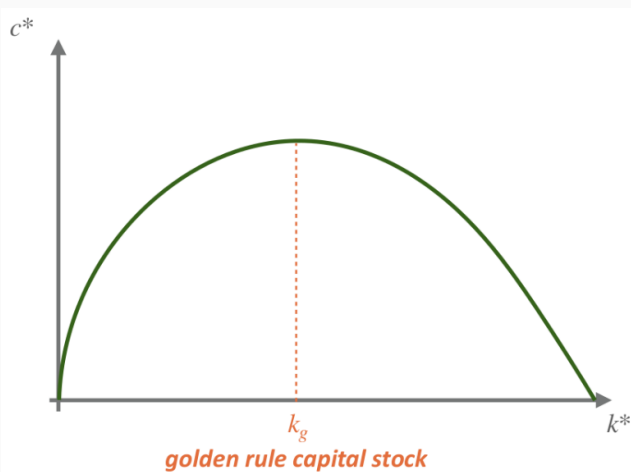
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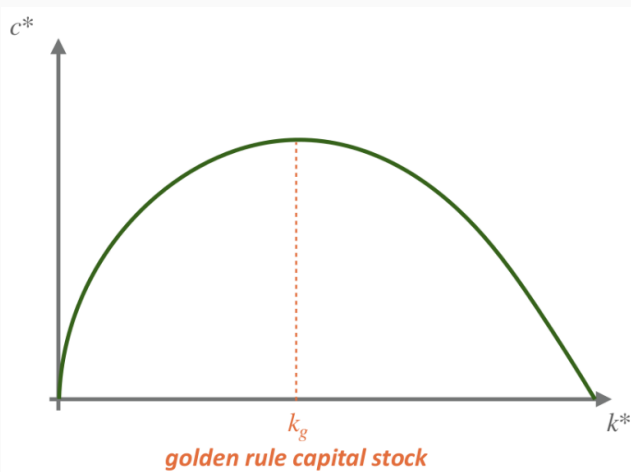
← *What is that curve?*

$$\tilde{c}^{ss} = (1 - \textcolor{brown}{s}) \cdot \left[ \tilde{k}^{ss}(\textcolor{brown}{s}) \right]^\alpha$$

*All values of  $\tilde{c}$  consistent with  $g_{\tilde{k}} = 0$*

# The “golden rule” to optimize steady-state consumption

(Phelps, 1961)

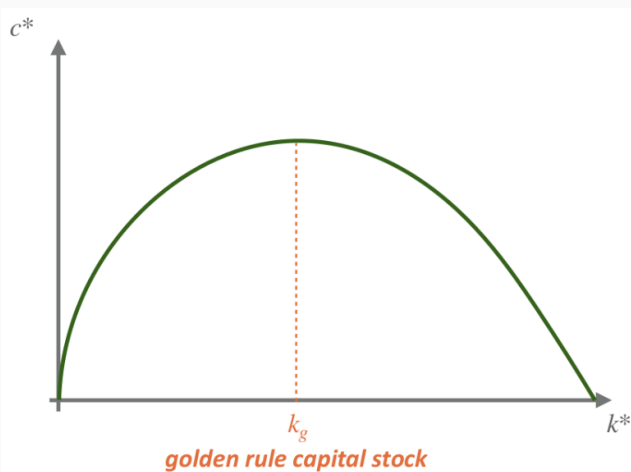


Solving  $\max_s \tilde{c}^{ss}$ :

$$\begin{aligned}\tilde{y}^{ss} &= (1-s) \frac{\partial \tilde{y}^{ss}}{\partial s} \\ \frac{s}{1-s} &= \frac{\partial \tilde{y}^{ss}}{\partial s} \frac{s}{\tilde{y}^{ss}} \\ \frac{s}{1-s} &= \frac{\alpha}{1-\alpha} \\ s^{GR} &= \alpha\end{aligned}$$

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*What's the intuition?*

## The Solow model in discrete time

As we switch into discrete time, let's simplify:  $L_t = 1$ ,  $A_t = 1 \ \forall t \rightarrow K_t = k_t = \tilde{k}_t$

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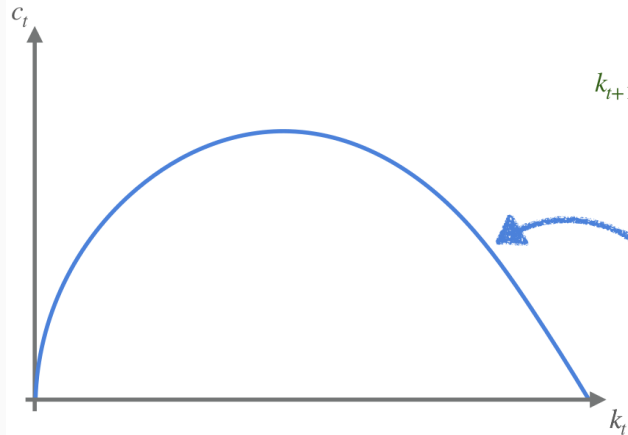
$$k_{t+1} = sF(k_t, 1) + (1 - \delta)k_t$$

but we can split that into...

$$k_{t+1} = [F(k_t, 1) - c_t] + (1 - \delta)k_t, \quad c_t = (1 - s)F(k_t, 1)$$

A steady state is a point where  $\frac{k_{t+1} - k_t}{\Delta t} = 0 \rightarrow k_{t+1} = k_t \rightarrow c_{t+1} = c_t$

## (Re)introducing the $c$ - $k$ phase diagram

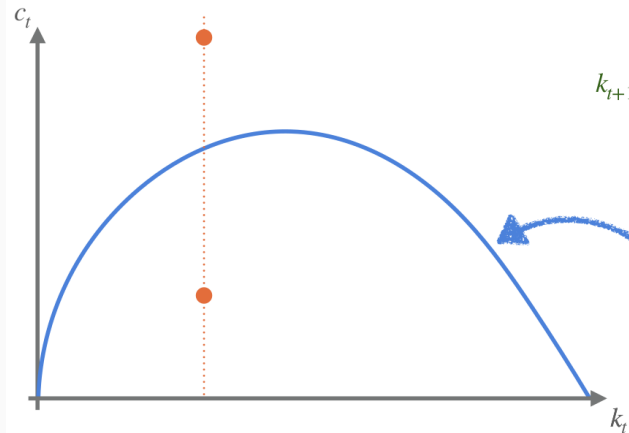


*the law of motion:*

$$k_{t+1} = F(k_t, 1) - c_t + (1 - \delta) k_t$$

*Q: How can we find the points for which  $k_{t+1} = k_t$ ?*

## (Re)introducing the $c$ - $k$ phase diagram



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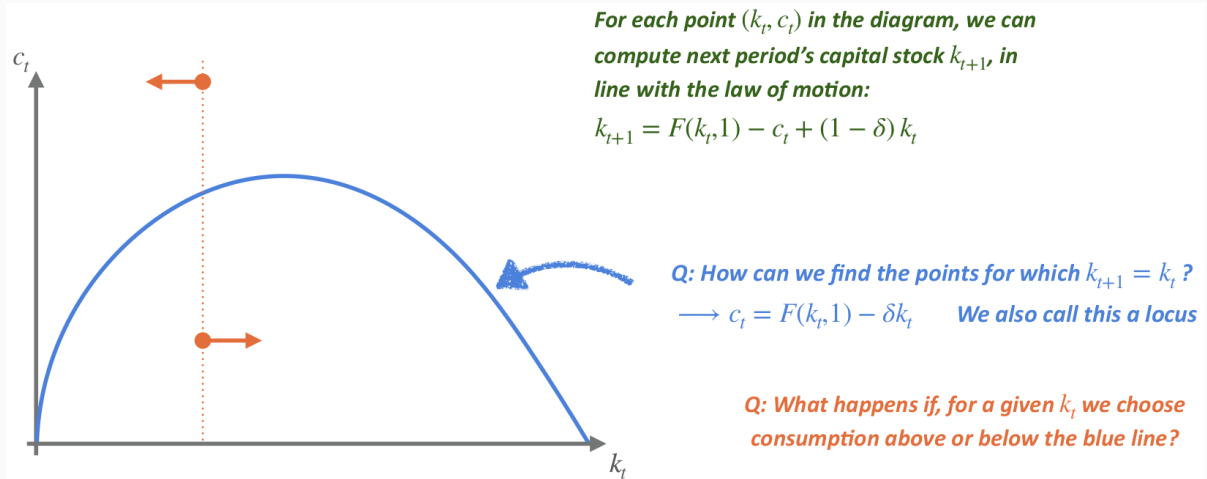
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*Q: How can we find the points for which  $k_{t+1} = k_t$ ?*

*$\longrightarrow c_t = F(k_t, 1) - \delta k_t$  We also call this a locus*

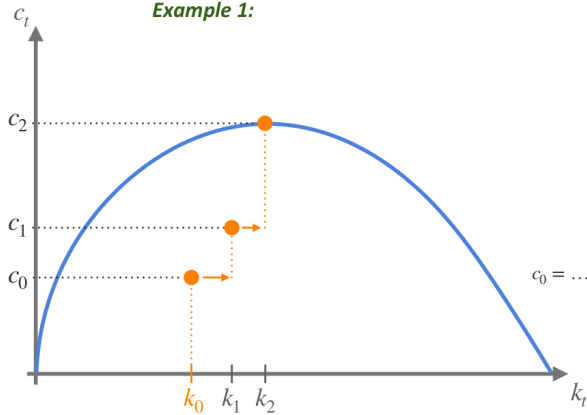
*Q: What happens if, for a given  $k_t$  we choose consumption above or below the blue line?*

## (Re)introducing the $c$ - $k$ phase diagram

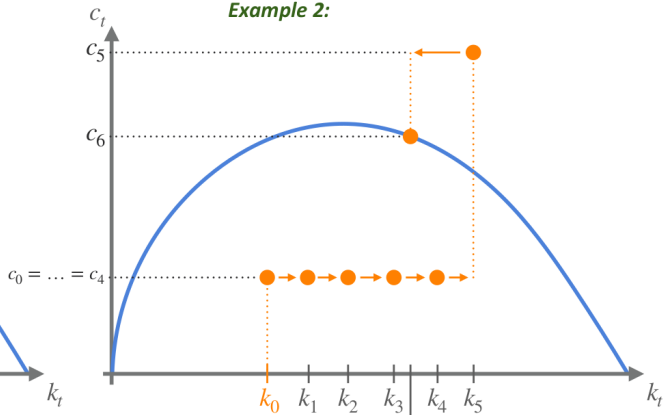


For any  $k_0$ , different consumption rules  $\rightarrow$  different transition paths

*Example 1:*

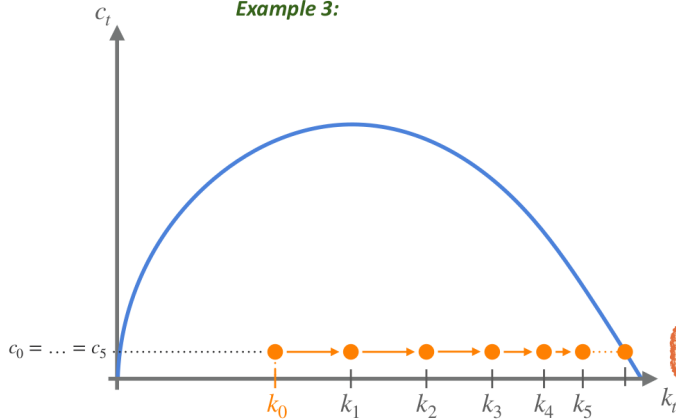


*Example 2:*



Some paths are clearly inferior to others...

*Example 3:*



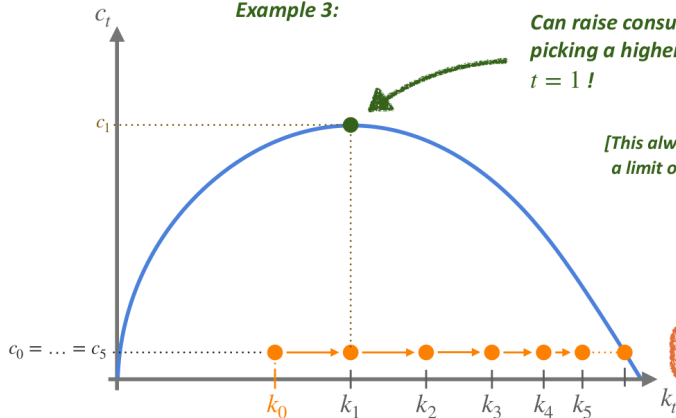
*Q: Why would this be a pretty inefficient path?*

# Some paths are clearly inferior to others...

*Example 3:*

*Can raise consumption everywhere by just picking a higher consumption starting at date  $t = 1$  !*

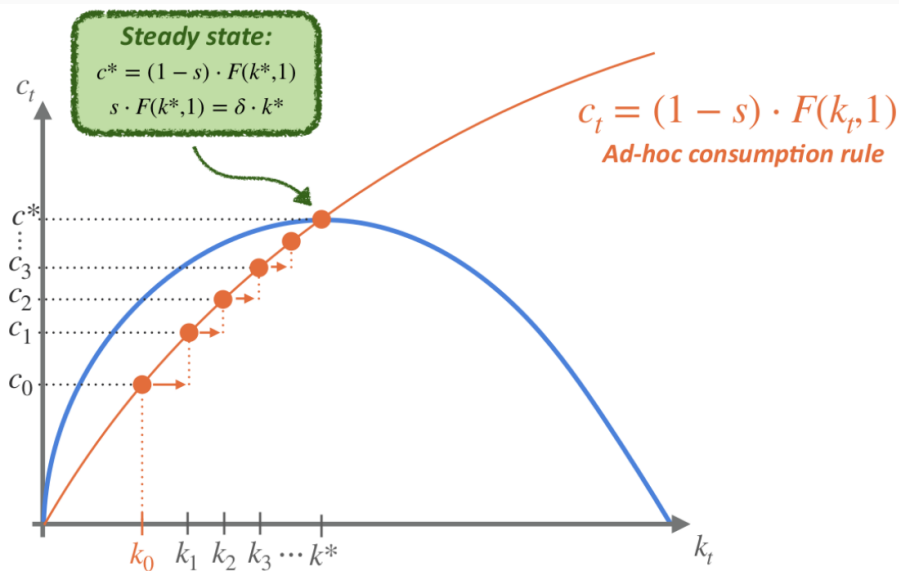
*[This always works if a path converges to a limit on the right side of the diagram]*



*Q: Why would this be a pretty inefficient path?*

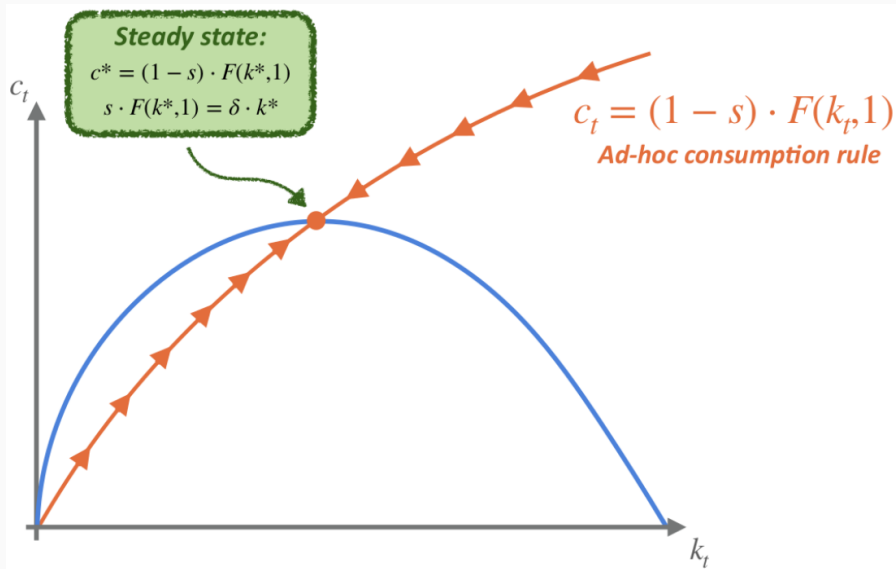
*[We call this dynamic inefficiency]*

## Solow consumption rule $\rightarrow$ just one particular path





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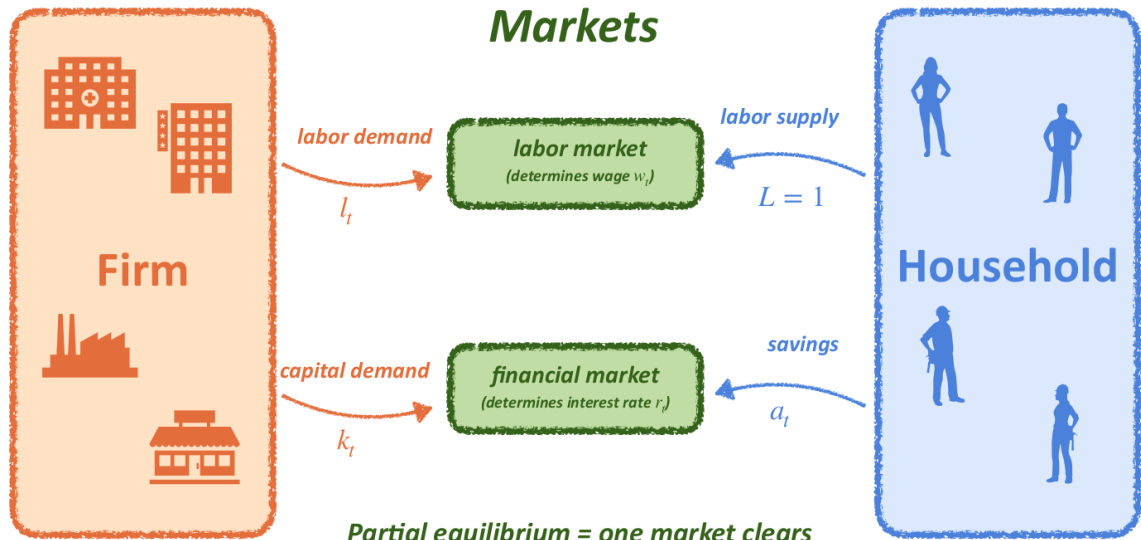


## The Ramsey-Cass-Koopmans model

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- The **Ramsey-Cass-Koopmans** model will be our first **fully modern** macro model
- **Main idea:** Want to replace the ad-hoc consumption rule of the Solow model...
  - ...with consumption behavior from a utility-maximizing  $\infty$ -horizon household
- Two approaches to set up this model
  - Describe as **general equilibrium**
    - Representative household & firm, interact thru markets *[will focus on this one!]*
  - Describe as **planning problem**
    - A “benevolent planner” maximizes household utility s.t. resource constraints
- *Note:* Both yield the same, so we say that the model is **first-best efficient**

# General equilibrium



# Household optimization problem

- Household earns **wage**  $w_t$  and accumulates a stock of **assets**  $a_t$  entering period  $t$ 
  - *last class*: income was denoted by  $Y_t$  and savings by  $S_t$
  - in general equilibrium, using  $w_t$  and  $a_t$  is somewhat more common

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which is characterized by the **Euler equation**

$$U'(c_t) = \beta(1 + r_{t+1})U'(c_{t+1}) \quad \text{for } t = 0, 1, 2, \dots$$

and **budget feasibility**

(recall from last class!)



# Firm optimization problem

- In a general equilibrium model, we also need to describe:
  - **what** the household saves in...
  - ...and **who** pays the wage
- Suppose there is a representative firm that, **each period**:
  - Rents capital  $k_t$  from the household at rate  $R_t = r_t + \delta$ 
    - compensate household for *opportunity cost* ( $r_t$ ) and *depreciation* ( $\delta$ )
    - *note*: what we called  $r$  in the Solow model would here be  $R_t$
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$$\max_{k_t, \ell_t} \underbrace{F(k_t, \ell_t)}_{\text{revenue}} - \overbrace{(r_t + \delta)k_t}^{\text{capital expenditures}} - \underbrace{w_t \ell_t}_{\text{wage payments}}$$

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$$\frac{\partial F(k_t, \ell_t)}{\partial k_t} \equiv F_k(k_t, \ell_t) = r_t + \delta$$
$$\frac{\partial F(k_t, \ell_t)}{\partial \ell_t} \equiv F_\ell(k_t, \ell_t) = w_t$$

# Defining equilibrium

A **competitive equilibrium** consists of quantities  $\{c_t, a_t, k_t, \ell_t\}$  and prices  $\{r_t, w_t\}$  s.t.

1. The household optimally picks  $\{c_t, a_t\}$  given  $\{r_t, w_t\}$

$$U'(c_t) = \beta(1 + r_{t+1})U'(c_{t+1}), \quad c_t + a_{t+1} = (1 + r_t)a_t + w_t \quad \text{for } t = 0, 1, 2, \dots$$

2. The firm optimally picks  $\{k_t, \ell_t\}$  given  $\{r_t, w_t\}$

$$F_k(k_t, \ell_t) = r_t + \delta, \quad F_\ell(k_t, \ell_t) = w_t$$

3. The asset market clears:  $k_t = a_t$

4. The labor market clears:  $\ell_t = 1$

5. The resource constraint holds: (check: already implied by #1–4)

$$k_{t+1} = F(k_t, 1) - c_t + (1 - \delta)k_t$$

# Ramsey-Cass-Koopmans vs. Solow

Both have **same** resource constraint...

$$k_{t+1} = F(k_t, 1) - c_t + (1 - \delta)k_t$$

... but **different** consumption rules:

$$c_t = (1 - s)F(k_t, 1)$$

vs.

$$U'(c_t) = \beta[1 + r_{t+1}]U'(c_{t+1})$$

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How can we **solve** this system? **Just like Solow!**

1. Characterize the **steady state**
2. Analyze transitions using  **$c$ - $k$  phase diagram**

# Steady state

A steady state is a pair  $\{c^{ss}, k^{ss}\}$  at which **optimally**...

$$c_t = c_{t+1} = c^{ss} \quad \rightarrow \quad U'(c^{ss}) = \beta [1 + F_k(k^{ss}, 1) - \delta] U'(c^{ss})$$

$$F_k(k^{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

$$k_t = k_{t+1} = k^{ss} \quad \rightarrow \quad k^{ss} = F(k^{ss}, 1) - c^{ss} + (1 - \delta)k^{ss}$$

$$c^{ss} = F(k^{ss}, 1) - \delta k^{ss}$$

*Can be shown:* optimal  $\{c_t, k_t\}$  **always converges** to  $\{c^{ss}, k^{ss}\}$

(Check: What is the relationship between  $\beta$  and  $r^{ss}$ ?)



## The “*modified* golden rule” to optimize the consumption *path*

Recall the **golden rule** ( $s^{\text{GR}} = \alpha$ ):

(Cobb-Douglas,  $L_t = 1$ ,  $A_t = 1$ )

$$k_{\text{Solow}}^{\text{ss}} = \left( \frac{s}{\delta + g_A + g_L} \right)^{\frac{1}{1-\alpha}} \rightarrow k^{\text{GR}} = \left( \frac{\alpha}{\delta} \right)^{\frac{1}{1-\alpha}}$$

But, from the Kaldor facts:  $s \approx \frac{1}{4} < \frac{1}{3} \approx \alpha \rightarrow$  **should we be saving more?**

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The **modified golden rule** says *not necessarily*:

$$F_k(k^{\text{ss}}, 1) = \frac{1}{\beta} - 1 + \delta \rightarrow k^{\text{MGR}} = \left( \frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}} < k^{\text{GR}}$$

The golden rule **didn't** account for impatience!

# What happens away from steady state?

Substitute the **resource constraint** into the **Euler equation**

$$U'(c_t) = \beta \left[ 1 + F_k(\underbrace{F(k_t, 1) - c_t + (1 - \delta)k_t}_{k_{t+1}: \text{resource constraint}}, 1) - \delta \right] U'(c_{t+1})$$

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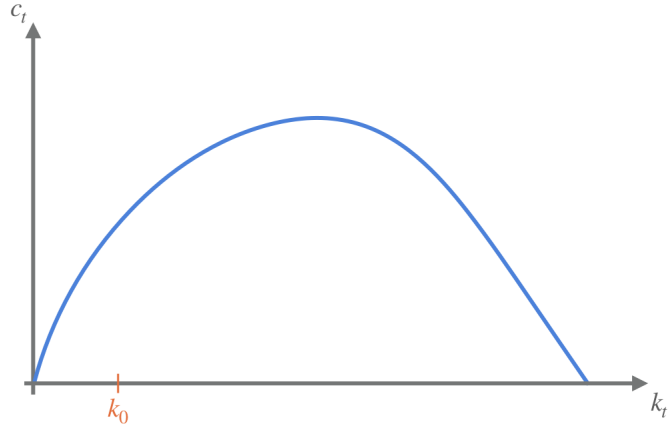
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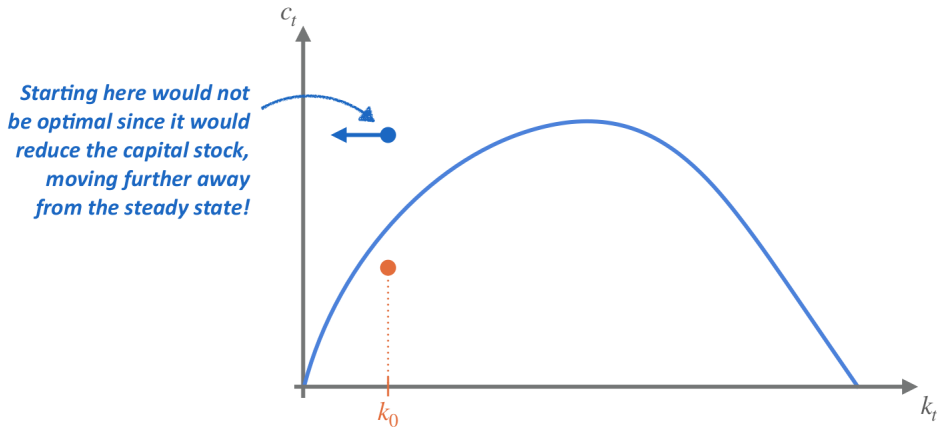
With **Cobb-Douglas** production and **log utility**...

$$c_{t+1} = \beta \left[ 1 + \alpha(k_t^\alpha - c_t + (1 - \delta)k_t)^{\alpha-1} - \delta \right] c_t$$

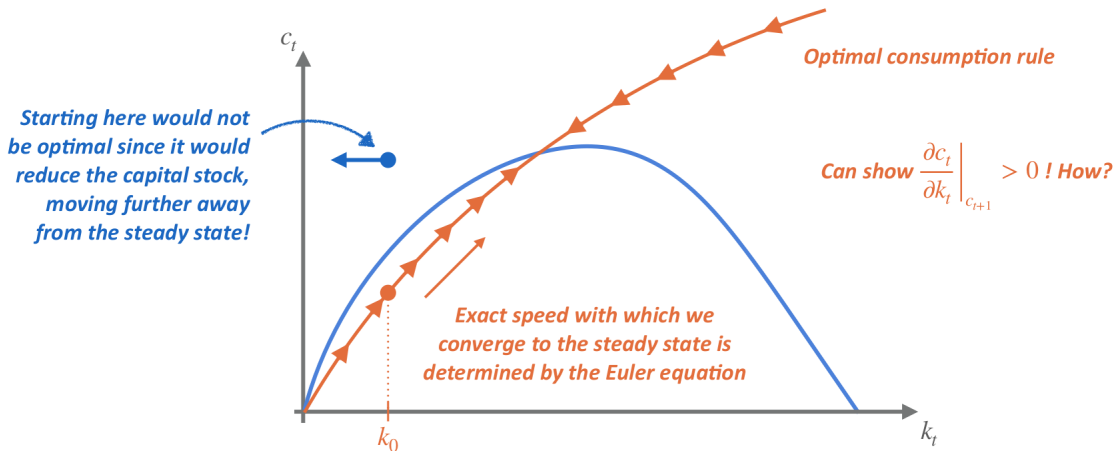
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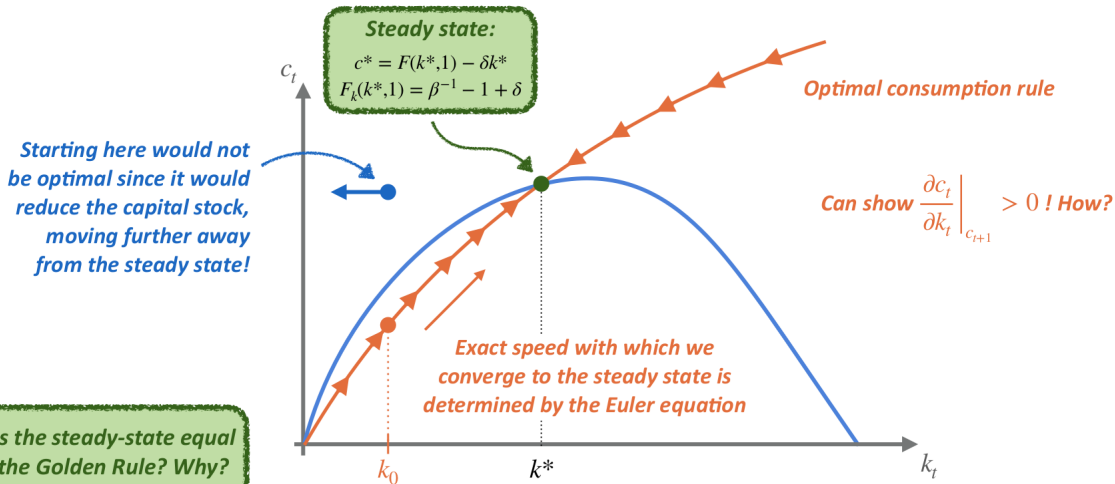


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Here's a special case we can solve in **closed-form**:

(Problem Set #3)

- Cobb-Douglas production function:  $F(k_t, 1) = k_t^\alpha$
- Log utility:  $U(c_t) = \ln(c_t)$
- Full depreciation:  $\delta = 1$  *(pretty unrealistic unless we use long time periods)*

Then the **optimal consumption rule** is given by...

$$c_t = (1 - \alpha\beta)k_t^\alpha$$

... which **looks exactly like Solow** just with **endogenous** savings rate  $s = \alpha\beta$

Let's study **two applications** of the Ramsey-Cass-Koopmans model:

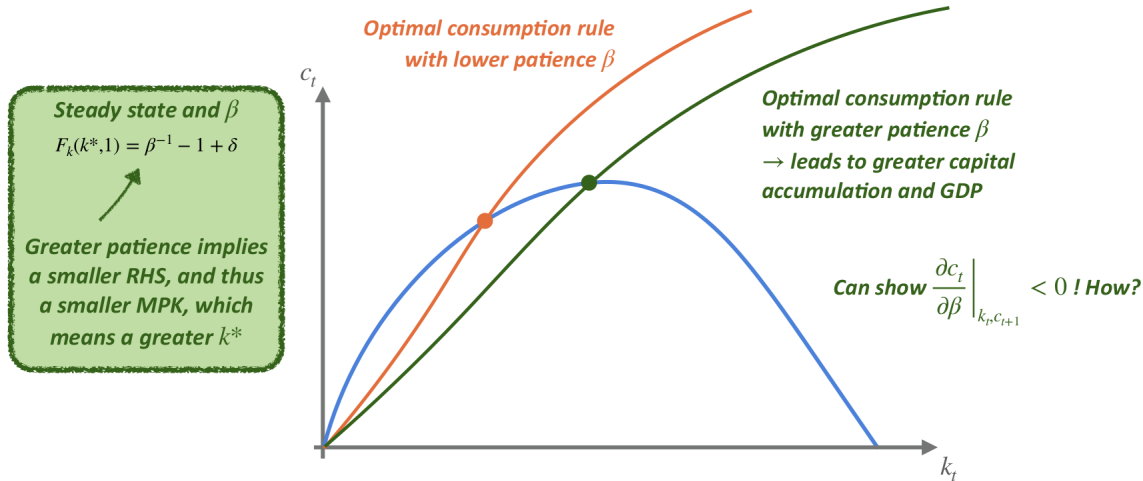
1. Compare two economies with different **discount factors**

- *idea*: Protestant culture should lead to greater patience
- *idea*: Aging economies should look like more patient economies

2. Introduce **capital taxes**

- *idea*: Economies with extractive institutions should end up saving less

## Application 1: Differences in discount factors



## Application 2: Introduce capital taxes

- Think of capital taxes as **reducing the return on saving** in the Euler equation

$$U'(c_t) = \beta[1 + r_{t+1}(\mathbf{1} - \tau)]U'(c_{t+1}) \rightarrow (\mathbf{1} - \tau) [F_k(k^{\text{ss}}, 1) - \delta] = \frac{1}{\beta} - 1$$

- capital taxes ( $\tau \uparrow$ ) have the **same effect** as less patience ( $\beta \downarrow$ )
- reduce capital accumulation and steady-state GDP per capita

# The neoclassical growth model (NGM)

$$F(K_t, A_t L_t) = K_t^\alpha (A_t L_t)^{1-\alpha}, \quad K_{t+1} = F(K_t, A_t L_t) - C_t + (1 - \delta)K_t$$

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... with **exogenous** savings (the **Solow** model)

- ad-hoc consumption rule  $\rightarrow$  savings depend on current output  $y_t$  only

... with **endogenous** savings (the **Ramsey-Cass-Koopmans** model)

- optimal consumption path  $\rightarrow$  savings depend on  $\{w_t, r_t, \tau_t\}, \beta, k_0, \dots$
- can reincorporate population ( $g_L$ ) and productivity ( $g_A$ ) growth

# How does the NGM answer our three organizing questions?

1. Why are we so rich and they so poor?
2. What is the engine of economic growth?
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*Next week: How do these fit the data?*

## Bonus: A better welfare metric?

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- *begs the question*: can we come up w/ a **better welfare metric** than just  $y_{it}$ ?  
say, one that incorporates...
  - optimal consumption-savings decisions?
  - optimal labor-leisure decisions?
  - mortality?
  - inequality?

*What proportion of consumption in the United States, given the US values of leisure, mortality, and inequality, would deliver **the same expected utility** as the values in country  $i$ ?*



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*Being a random person in country  $i$  (say, France) = Being a random person in the US with your consumption scaled by  $\lambda_i$ .*

Suppose everyone in the world has the **same preferences**:

$$U(c, \ell) = \bar{u} + \ln(c) + v(\ell)$$

with, for each country  $i$ , no discounting ( $\beta = 1$ ), no consumption growth ( $g_c = 0$ ), constant leisure ( $\ell_i$ ), known life expectancy ( $e_i$ ), and consumption distributed  $\text{LogNormal}(c_i, \sigma_i^c)$  across people in  $i$ .

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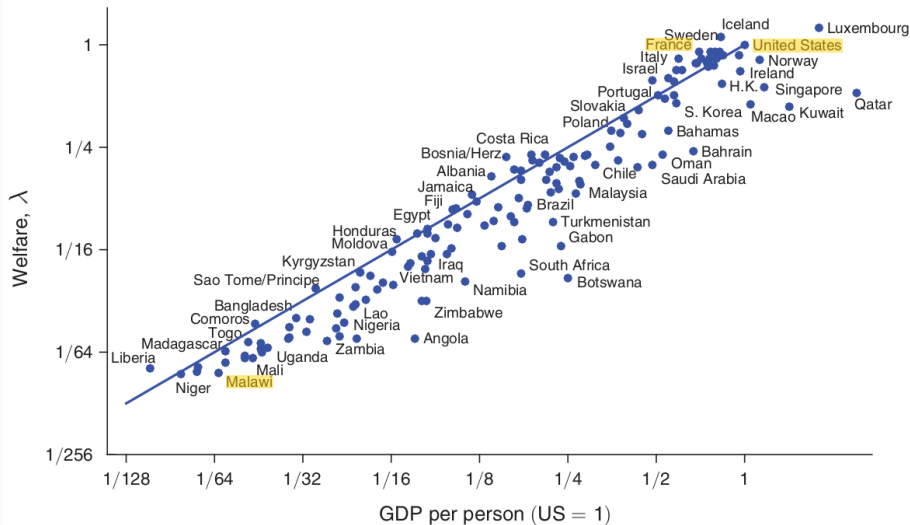
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...and the (log) **consumption-equivalent welfare** in country  $i$  *relative to* the US is

$$\begin{aligned} \ln \lambda_i &= \frac{e_i - e_{\text{US}}}{e_{\text{US}}} [\bar{u} + \ln(c_i) + v(\ell_i) - 0.5\sigma_i^c] && \text{(Life expectancy)} \\ &+ \ln c_i - \ln c_{\text{US}} && \text{(Consumption)} \\ &+ v(\ell_i) - v(\ell_{\text{US}}) && \text{(Leisure)} \\ &- 0.5(\sigma_i^c - \sigma_{\text{US}}^c) && \text{(Inequality)} \end{aligned}$$

Panel A. Welfare and income are highly correlated at 0.96



Panel B. But this masks substantial variation in the ratio of  $\lambda$  to GDP per capita. The mean absolute deviation from unity is about 27%

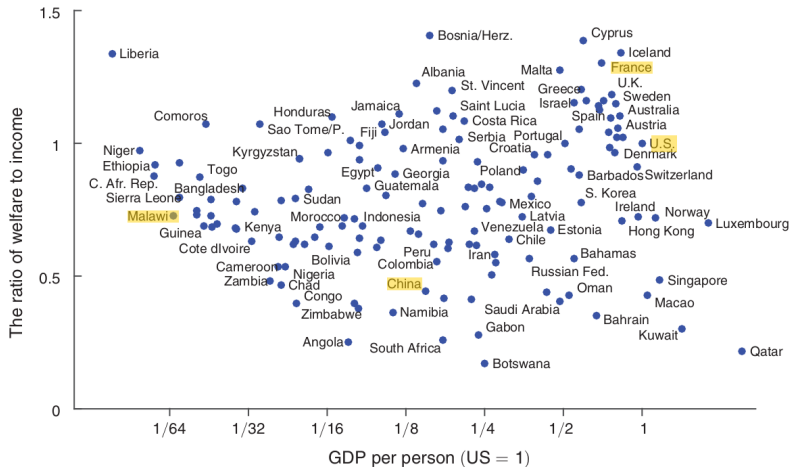


FIGURE 7. WELFARE USING MACRO DATA, 2007

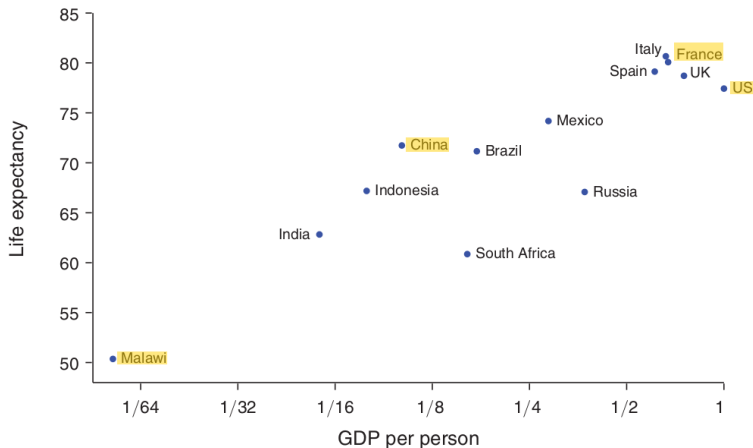


FIGURE 4. LIFE EXPECTANCY

*Note:* Life expectancy at birth in each country is measured as the sum over all ages of the probability of surviving to each age, using life tables from the World Health Organization.



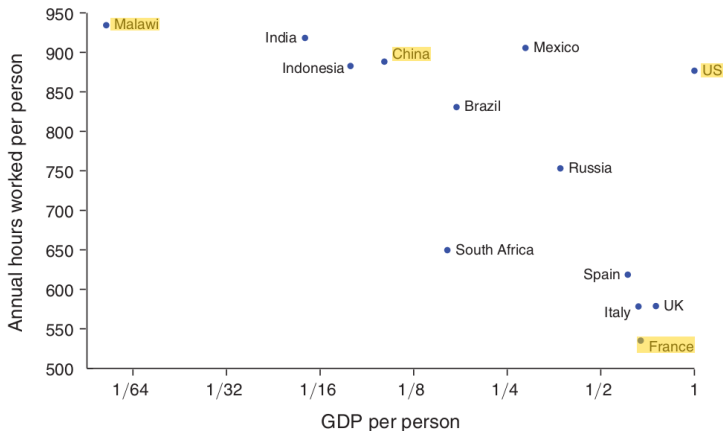


FIGURE 2. ANNUAL HOURS WORKED ACROSS COUNTRIES

*Notes:* The measure shown here of annual hours worked per capita is computed from the household surveys noted in Table 1, using survey-specific sampling weights and US survival rates across ages as in equation (16), with no time discounting.

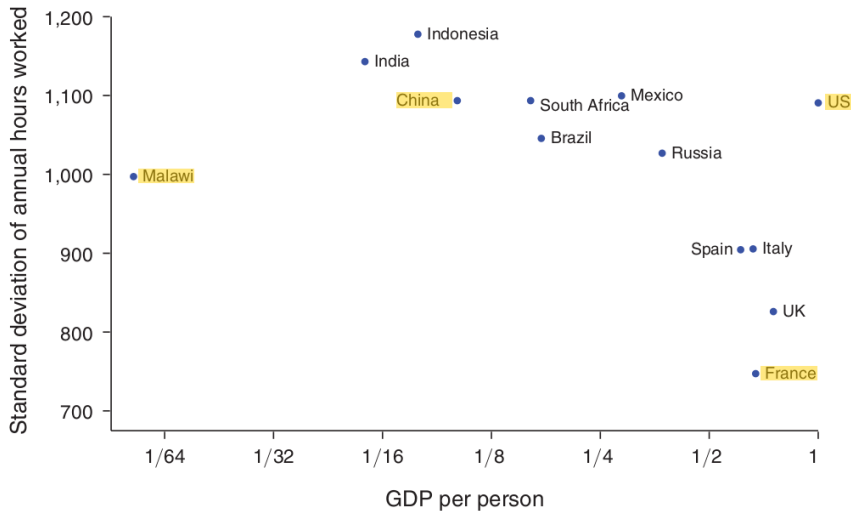


FIGURE 3. INEQUALITY IN ANNUAL HOURS WORKED

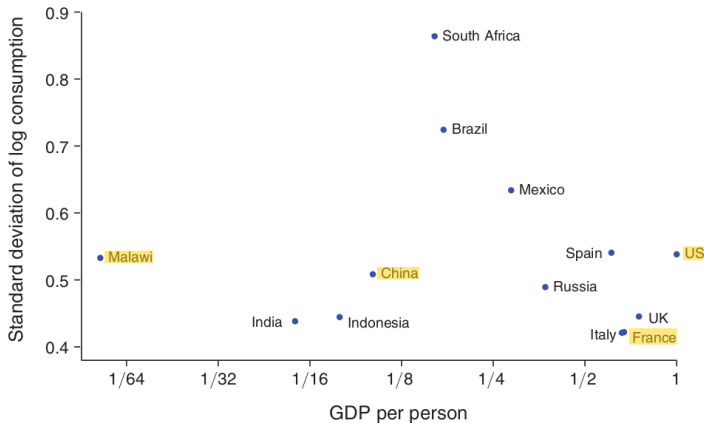


FIGURE 1. WITHIN-COUNTRY INEQUALITY

*Notes:* The standard deviation of log consumption within each economy is measured from the household surveys listed in Table 1. We use survey-specific sampling weights and US survival rates across ages using an analog of equation (17), with no discounting or growth.

That was for *levels*, but it's likewise for *growth*

(Jones and Klenow, 2016)

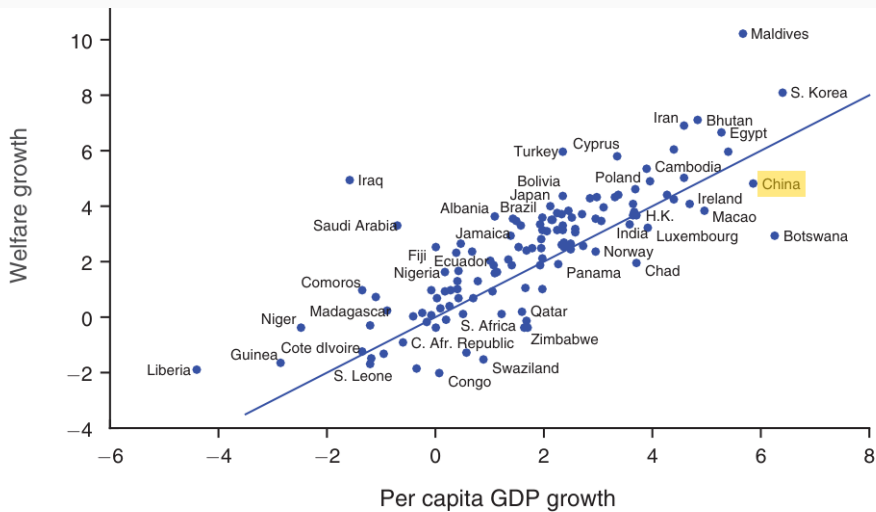
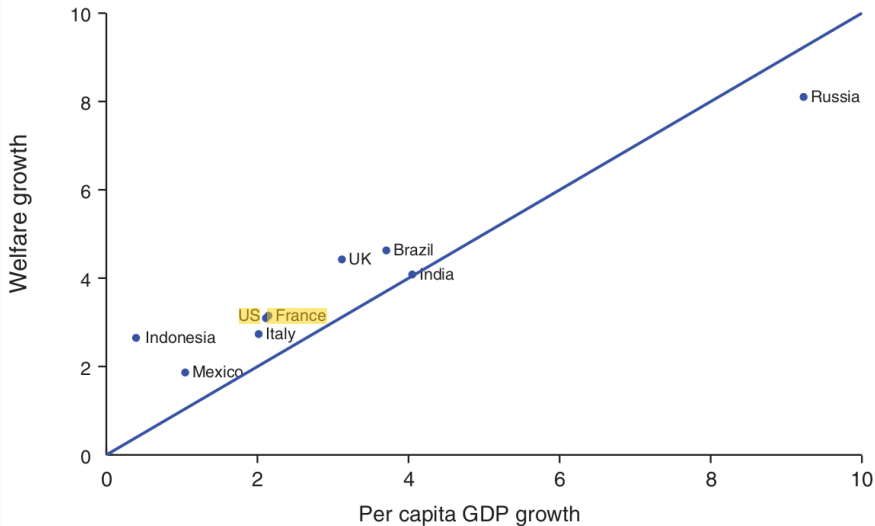
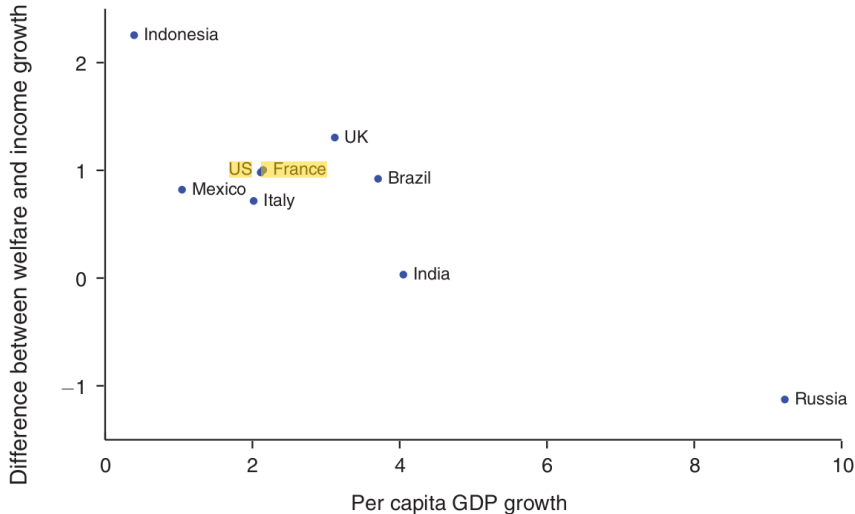


FIGURE 8. WELFARE AND INCOME GROWTH, 1980–2007 (Percent)

Panel A. The correlation between welfare growth and income growth is 0.97



Panel B. The median absolute value of the difference between welfare and income growth is 0.95 percentage points



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