ECON 164: Theory of Economic Growth

Week 5: AK Models

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    2. **misspecification**: maybe we're using the wrong $\boldsymbol{F}_{it}(\cdot, \cdot)$, omitting inputs, . . .
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  - **Next few lectures**: models of "endogenous growth"

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- anchor to Solow, but everything we say applies to Ramsey-Cass-Koopmans too

The AK model: A limiting case of the Solow model

What happens in the **Solow model**. . . (with *constant* Hicks-neutral productivity)

$$Y_t = AK_t^{\alpha}(L_t)^{1-\alpha}, \qquad \dot{K}_t = sY_t - \delta K_t$$

. . . if we set $\alpha = 1$?

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. . . if we set $\alpha = 1$? The model becomes. . .

$$Y_t = AK_t, \qquad \dot{K}_t = sY_t - \delta K_t$$

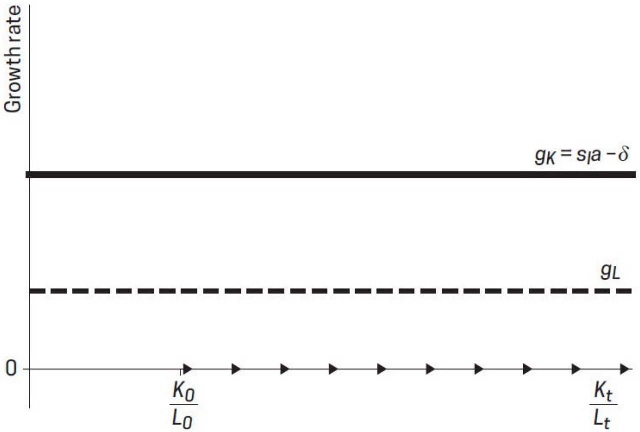
. . . so the growth rate of capital becomes

$$g_K = \frac{\dot{K}_t}{K_t} = sA - \delta$$

(Problem Set #2, Question 1B)

Dynamics when $\alpha = 1$: AK model generates sustained growth thru K

Figure 11.1 Dynamics of the *AK* Model



NOTE: The growth rate of capital, g_K, does not depend on the ratio K_t/L_t, so the two curves never cross and there is permanent growth in K_t/L_t over time.

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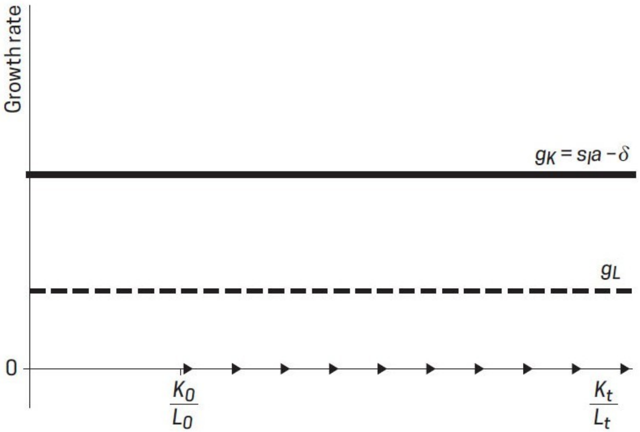

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Growth rate of output per capita:

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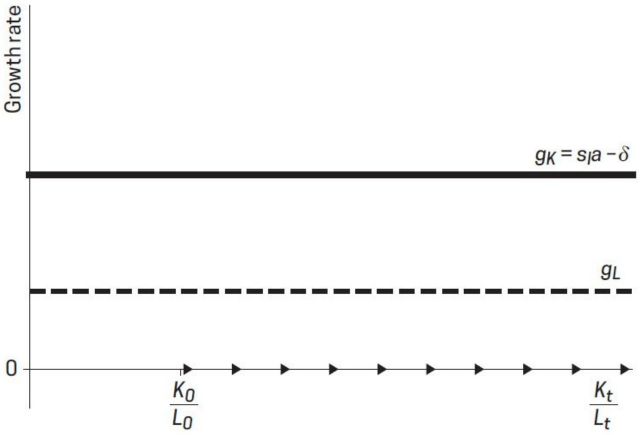

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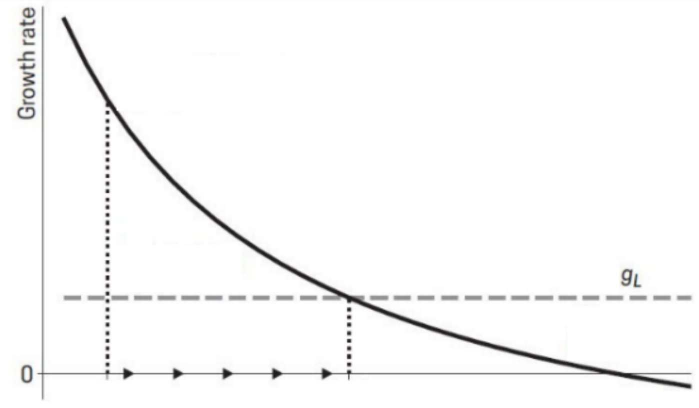
Growth rate of output per capita:

$$\boldsymbol{g_y} = g_K - g_L = \boldsymbol{sA - \delta - g_L}$$

Some observations:

- $\boldsymbol{g_y}$ is **independent** of k_t
- $\boldsymbol{g_y}$ is **positive** and **persistent** even w/ $g_A = 0$
- $\boldsymbol{g_y}$ is affected **permanently** by a change in policy ($\boldsymbol{s}$)

Intuition: It's all about (avoiding) diminishing returns. . .



For $\alpha < 1$ . . .

$$\dot{k} = sAk^{\alpha} - (\delta + g_L)k$$

$$\downarrow$$

$$g_y = \alpha(sAk^{\alpha-1} - \delta - g_L)$$

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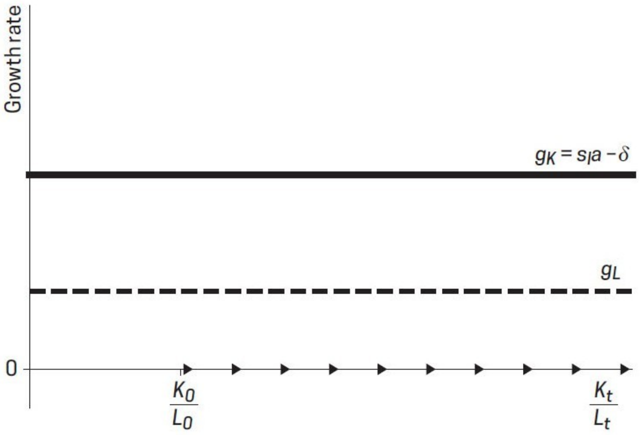

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. . . but with $\alpha = 1$:

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…somewhere in the model

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In **general**, there must be some variable X_t governed by a **linear differential equation**

$$\dot{X}_t = g_X X_t$$

(*why isn't* $\dot{L}_t = g_L L_t$ *enough then?*)

- Recall the **production function** with human capital (but *constant* Hicks-neutral A)

$$Y_t = AK_t^{\alpha}(uh_tL_t)^{1-\alpha}$$

where we introduce $u \equiv$ **exogenous share** of labor used for production

The Uzawa-Lucas model (Uzawa, 1965; Lucas, 1988)

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- Accumulating **physical capital** happens as it does in Solow...

$$\dot{K}_t = sY_t - \delta K_t$$

- ...but now we explicitly accumulate **human capital**, too:

$$\dot{h}_t = (1-u)h_t$$

The Uzawa-Lucas model

On the one hand, this is a cousin of the **AK model** → same long-run dynamics

- $\dot{h}_t$ is linear in h_t
- $\dot{K}_t$ is linear in $\{K_t, h_t\}$ because $Y_t = F(K_t, h_t, L_t)$ is CRS in that tuple

$$\begin{aligned} \dot{K}_t(\lambda K_t, \lambda h_t) &= sF(\lambda K_t, \lambda h_t, L_t) - \delta \lambda K_t \\ &= s\lambda F(K_t, h_t, L_t) - \delta \lambda K_t = \lambda \dot{K}_t(K_t, h_t) \end{aligned}$$

On the other, it's still a cousin of the **Solow model** → same transitional dynamics

- h_t acts like **labor-augmenting** productivity
- balanced growth path (BGP) requires **constant** $\hat{k}_t = \frac{K_t}{h_t L_t}$ . . .
- . . . so both s and u affect the level, but only u affects the growth rate of y_t

The pure externality model (Romer, 1986)

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- Suppose individual firms treat B_t as **exogenous**, but in reality...
  - **learning-by-doing** occurs as a by-product of each firm's net investment (evidence for shipbuilding, solar panels, ...)
  - that new knowledge **spills over** immediately to all other firms

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- This maintains assumption of **perfect competition** → vs. next week

Case closed? Not quite. . .

We can generate **positive, long-run growth** with AK models, so we're done, right?

1. growth vs. level effects
2. scale effects
3. what do firms actually do?

Growth vs. level effects

- Should increased s raise growth ***forever***? Call this a **growth effect**...
  - AK models imply it does: $g_y = \boldsymbol{s}A - \delta - g_L$ (Uzawa-Lucas: think about $\boldsymbol{u}$)
  - NGM says it doesn't, only a **level effect**: $\ln y^{\mathsf{BGP}} = g_A t + \ln A_0 + \frac{\alpha}{1-\alpha} \ln \tilde{k}^{\mathsf{ss}}(\boldsymbol{s})$
- Generically, **growth effect** $\approx$ **level effect w/ long transition**, but not vice versa
  - so level effect is **more empirically flexible**...
  - ...but growth effect is often **theoretically cleaner**

[advanced reading: McGrattan (1998)]

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- A large literature worked out models to eliminate these scale effects...
  - Segerstrom (1998); Young (1998); Howitt (1999); Jones (1999); Peretto (2018)
  - Uzawa-Lucas avoids it b/c what matters is **average**, not **total**, human capital
- ...but maybe we just need to think harder about the right **spatial scale**:
  - maybe it's the **whole world** in the very long run (Kremer, 1993)
  - maybe it's just **local labor markets**

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How do we match these **micro facts** in a **macro model**?

That's what we cover the next two weeks!

What *is* endogenous growth?

- **our running definition**: long-run growth determined by modeled choices (Slide 3)
- **the 1980–90s answer**: permanent changes in policies have **permanent effects** on an economy's long-run growth rate
- **the modern answer**: the endogenous outcome, **permanent** or **transitory**, of an economy in which profit-seeking individuals who are allowed to earn rents on the fruits of their labors search for newer and better ideas (Jones and Vollrath, 2024, p.241)

when only transitory, prefer to call this **semi-endogenous**

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- but, at best, innovation is **accidental** $\to$ next two weeks

References

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