ECON 164: Theory of Economic Growth

Week 9B: Review

Levi Crews

Winter 2026

Today

1. **zoom out**: how have we answered our three organizing questions?

2. **review our common toolkit**
   - Cobb-Douglas production function w/ labor-augmenting technology
   - factor accumulation: $\dot{K}$, $\dot{L}$, $\dot{A}$
   - steady states, phase diagrams, BGPs, ...

3. **practice!** esp. comparative statics

Three organizing questions

1. **Why are we so rich and they so poor?**

2. **What is the engine of economic growth?**

3. **How do "growth miracles" happen?**

Three organizing questions

1. **Why are we so rich and they so poor?**

   $\rightarrow$ differences in $\boldsymbol{s_I}$ $(\beta, \tau, \ldots)$, $\boldsymbol{g_L}$, $\boldsymbol{A_0}$ (or $\boldsymbol{D_0}$, $\boldsymbol{B_0}$, ...)

2. **What is the engine of economic growth?**

3. **How do "growth miracles" happen?**

Three organizing questions

1. **Why are we so rich and they so poor?**

   $\rightarrow$ differences in $\boldsymbol{s_I}$ $(\beta, \tau, \ldots)$, $\boldsymbol{g_L}$, $\boldsymbol{A_0}$ (or $\boldsymbol{D_0}$, $\boldsymbol{B_0}$, ...)

2. **What is the engine of economic growth?**

   $\rightarrow$ **technology!** a few frontier countries invent new ideas $(\dot{\boldsymbol{A}})$ using people $(\boldsymbol{g_L})$

   ... while the rest adopt $(\dot{\boldsymbol{D}})$ or import them $(\boldsymbol{M})$

3. **How do "growth miracles" happen?**

Three organizing questions

1. **Why are we so rich and they so poor?**

   $\rightarrow$ differences in $\boldsymbol{s_I}$ $(\beta, \tau, \ldots)$, $\boldsymbol{g_L}$, $\boldsymbol{A_0}$ (or $\boldsymbol{D_0}$, $\boldsymbol{B_0}$, ...)

2. **What is the engine of economic growth?**

   $\rightarrow$ **technology!** a few frontier countries invent new ideas ($\dot{\boldsymbol{A}}$) using people ($\boldsymbol{g_L}$)

   ... while the rest adopt ($\dot{\boldsymbol{D}}$) or import them ($\boldsymbol{M}$)

3. **How do "growth miracles" happen?**

   $\rightarrow$ transitional growth from **physical capital** as in the NGM...

   ... and transitional growth from **technology** through R&D, adoption, and trade

Our approach: The cosmologists of economics

- one universe, one global economy $\rightarrow$ **no controlled experiments**
- instead a **back-and-forth between data and models**

  ...often the most important constraint on a new theory is not that it should survive this or that new experimental test, but that it should agree with the body of past observations, as crystallized in former theories. ...The wonderful thing is that the need to preserve successes of the past is not only a constraint, but also a guide.

  Weinberg (2018, Ch. 24)

Our approach: The cosmologists of economics

- one universe, one global economy $\rightarrow$ **no controlled experiments**
- instead a **back-and-forth between data and models**

  . . . often the most important constraint on a new theory is not that it should survive this or that new experimental test, but that it should agree with the body of past observations, as crystallized in former theories. . . . The wonderful thing is that the need to preserve successes of the past is not only a constraint, but also a guide.

  Weinberg (2018, Ch. 24)

- A **model** is a **mathematical representation** of some aspect of the economy. . .
  - agents' objectives, constraints
  - market structure, feasibility conditions
  - parameters, exogenous variables

  . . . that determines how we **interpret data** and **evaluate policy**

Two building blocks of every model

1. ***How do economies produce?***

2. ***How do economies accumulate factors of production?***

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} (A_t L_t)^{1-\alpha}$$

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} (A_t L_t)^{1-\alpha}$$

2. **vanilla AK model**

$$Y_t = \boldsymbol{K_t}^{\alpha} (A_t L_t)^{1-\alpha} \quad \text{with} \quad \alpha = 1$$

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

2. **vanilla AK model** (with constant Hicks-neutral A)

$$Y_t = A\boldsymbol{K_t}$$

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

2. **vanilla AK model** (with constant Hicks-neutral A)

$$Y_t = A\boldsymbol{K_t}$$

3. **semi-endogenous growth model** (Romer + Schumpeterian)

$$Y_t = \boldsymbol{K_t}^{\alpha} \left(\boldsymbol{A_t} L_{Yt}\right)^{1-\alpha}$$

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} (A_t L_t)^{1-\alpha}$$

2. **vanilla AK model** (with constant Hicks-neutral A)

$$Y_t = A\boldsymbol{K_t}$$

3. **semi-endogenous growth model** (Romer + Schumpeterian)

$$Y_t = \boldsymbol{K_t}^{\alpha} (\boldsymbol{A_t} L_{Yt})^{1-\alpha}$$

4. **technology diffusion model** (w/o imports)

$$Y_t = \boldsymbol{K_t}^{\alpha} (\boldsymbol{D_t} L_t)^{1-\alpha}$$

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} (A_t L_t)^{1-\alpha}$$

2. **vanilla AK model** (with constant Hicks-neutral A)

$$Y_t = A\boldsymbol{K_t}$$

3. **semi-endogenous growth model** (Romer + Schumpeterian)

$$Y_t = \boldsymbol{K_t}^{\alpha} (\boldsymbol{A_t} L_{Yt})^{1-\alpha}$$

4. **technology diffusion model** (w/ imports)

$$Y_t = \boldsymbol{K_t}^{\alpha} ((\boldsymbol{D_t} + \boldsymbol{M_t}) L_t)^{1-\alpha}$$

The production function

1. **neoclassical growth model** (Solow + Ramsey-Cass-Koopmans)

$$Y_t = \boldsymbol{K_t}^{\alpha} (A_t L_t)^{1-\alpha}$$

2. **vanilla AK model** (with constant Hicks-neutral A)

$$Y_t = A\boldsymbol{K_t}$$

3. **semi-endogenous growth model** (Romer + Schumpeterian)

$$Y_t = \boldsymbol{K_t}^{\alpha} (\boldsymbol{A_t} L_{Yt})^{1-\alpha}$$

4. **technology diffusion model** (w/ imports)

$$Y_t = \boldsymbol{K_t}^{\alpha} ((\boldsymbol{D_t} + \boldsymbol{M_t}) L_t)^{1-\alpha}$$

5. **natural resources model** (w/ energy and land)

$$Y_t = \boldsymbol{K_t}^{\alpha} (\boldsymbol{B_t} L_t)^{1-\alpha}$$

Sometimes we...

- **add human capital**:

$$Y_t = K_t^{\alpha} \left(A_t h_t L_t\right)^{1-\alpha}$$

- **move around the productivity term**:
  - Hicks-neutral vs. Harrod-neutral
  - Malthusian model

$$Y_t = X^{\beta}(B_t L_t)^{1-\beta} \equiv X^{\beta}\left(A_t^{\frac{\beta}{1-\beta}} L_t\right)^{1-\beta} = (A_t X)^{\beta} L_t^{1-\beta}$$

Sometimes we...

- **add human capital**:

$$Y_t = K_t^{\alpha} \left(A_t h_t L_t\right)^{1-\alpha}$$

- **move around the productivity term**:
  - Hicks-neutral vs. Harrod-neutral
  - Malthusian model

$$Y_t = X^{\beta}(B_t L_t)^{1-\beta} \equiv X^{\beta}\left(A_t^{\frac{\beta}{1-\beta}} L_t\right)^{1-\beta} = (A_t X)^{\beta} L_t^{1-\beta}$$

...but the nuts-and-bolts are the same!

Two building blocks of every model

1. ***How do economies produce?***

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

2. ***How do economies accumulate factors of production?***

The stage we've set

Growth can either be from...

- getting more K
- getting more L
- improving A

Accumulating factors

1. **neoclassical growth model**

$$\dot{K}_t = \boldsymbol{s_I} Y_t - \delta K_t$$

Accumulating factors

1. **neoclassical growth model** $\rightarrow$ *but also every subsequent model*

$$\dot{K}_t = \boldsymbol{s_I} Y_t - \delta K_t$$

Accumulating factors

1. **neoclassical growth model** $\rightarrow$ *but also every subsequent model*

$$\dot{K}_t = \boldsymbol{s_I} Y_t - \delta K_t$$

2. **semi-endogenous growth model**

$$\dot{A}_t = \theta(\boldsymbol{s_R} L_t)^\lambda A_t^\phi$$

Accumulating factors

1. **neoclassical growth model** $\longrightarrow$ *but also every subsequent model*

$$\dot{K}_t = \boldsymbol{s_I} Y_t - \delta K_t$$

2. **semi-endogenous growth model**

$$\dot{A}_t = \theta(\boldsymbol{s_R} L_t)^{\lambda} A_t^{\phi}$$

3. **technology diffusion model**

$$\dot{D}_t = \psi h A_t^{\zeta} D_t^{1-\zeta}$$

Accumulating factors

1. **neoclassical growth model** $\rightarrow$ *but also every subsequent model*

$$\dot{K}_t = \boldsymbol{s_I} Y_t - \delta K_t$$

2. **semi-endogenous growth model**

$$\dot{A}_t = \theta(\boldsymbol{s_R} L_t)^\lambda A_t^\phi$$

3. **technology diffusion model**

$$\dot{D}_t = \psi h A_t^\zeta D_t^{1-\zeta}$$

4. **natural resources model**

$$\dot{R}_t = -\boldsymbol{s_E} R_t$$

Two building blocks of every model

1. ***How do economies produce?***

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

2. ***How do economies accumulate factors of production?***

$$\dot{K}_t = s_I Y_t - \delta K_t$$
$$\dot{L}_t = g_L L_t$$
$$\dot{A}_t = \dots$$

. . . depends on the particular model!

Solving the model

What does it mean to "solve" the model?

Given...

- initial levels $\{K_0, A_0, L_0\}$

in class: ***with phase diagrams!***

- parameters $\{\alpha, \delta, g_L, \ldots\}$
- equations of the model

...we obtain a **time path** for the **endogenous** variables $\{K_t, Y_t, \ldots\}$ expressed in terms of the **exogenous** variables $\{L_t, \ldots\}$, parameters, and initial conditions.

What we *really* want to solve for: GDP per capita

GDP per capita $= y_t$

What we *really* want to solve for: GDP per capita

$$\text{GDP per capita} = \boldsymbol{y_t} \equiv \frac{Y_t}{L_t} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t$$

What we *really* want to solve for: GDP per capita

$$\text{GDP per capita} = \boldsymbol{y_t} \equiv \frac{Y_t}{L_t} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t \equiv \boldsymbol{\tilde{k}_t^{\alpha} A_t}$$

Growth rates: “take logs and derivatives”

For any variable x, let $\boldsymbol{g_x = \frac{d \ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

Growth rates: “take logs and derivatives”

For any variable x, let $\boldsymbol{g_x = \frac{d \ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

- often we plot $\ln x$ against time t...
- ...because then the slope ($\frac{d \ln x}{dt}$) *is* the growth rate

Growth rates: "take logs and derivatives"

For any variable x, let $\boldsymbol{g_x = \frac{d\ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

- often we plot $\ln x$ against time t...
- ...because then the slope ($\frac{d\ln x}{dt}$) *is* the growth rate

Let's take **logs** then **derivatives** w.r.t. t of the GDP per capita equation:

$$\ln y_t = \alpha \ln \tilde{k}_t + \ln A_t$$

Growth rates: "take logs and derivatives"

For any variable x, let $\boldsymbol{g_x = \frac{d \ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

- often we plot $\ln x$ against time t...
- ...because then the slope ($\frac{d \ln x}{dt}$) *is* the growth rate

Let's take **logs** then **derivatives** w.r.t. t of the GDP per capita equation:

$$\begin{aligned}
\ln y_t &= \alpha \ln \tilde{k}_t + \ln A_t \\
&= \alpha \left(\ln K_t - \ln A_t - \ln L_t \right) + \ln A_t
\end{aligned}$$

Growth rates: "take logs and derivatives"

For any variable x, let $\boldsymbol{g_x = \frac{d \ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

- often we plot $\ln x$ against time t...
- ...because then the slope ($\frac{d \ln x}{dt}$) *is* the growth rate

Let's take **logs** then **derivatives** w.r.t. t of the GDP per capita equation:

$$
\begin{aligned}
\ln y_t &= \alpha \ln \tilde{k}_t + \ln A_t \\
&= \alpha \left(\ln K_t - \ln A_t - \ln L_t \right) + \ln A_t \\
\frac{d \ln y_t}{dt} &= \alpha \left(\frac{d \ln K_t}{dt} - \frac{d \ln A_t}{dt} - \frac{d \ln L_t}{dt} \right) + \frac{d \ln A_t}{dt}
\end{aligned}
$$

Growth rates: “take logs and derivatives”

For any variable x, let $\boldsymbol{g_x = \frac{d \ln x}{dt} = \frac{dx/dt}{x} \equiv \frac{\dot{x}}{x}}$ be its growth rate

- often we plot $\ln x$ against time t...
- ...because then the slope $\left(\frac{d \ln x}{dt}\right)$ *is* the growth rate

Let's take **logs** then **derivatives** w.r.t. t of the GDP per capita equation:

$$
\begin{aligned}
\ln y_t &= \alpha \ln \tilde{k}_t + \ln A_t \\
&= \alpha \left(\ln K_t - \ln A_t - \ln L_t\right) + \ln A_t \\
\frac{d \ln y_t}{dt} &= \alpha \left(\frac{d \ln K_t}{dt} - \frac{d \ln A_t}{dt} - \frac{d \ln L_t}{dt}\right) + \frac{d \ln A_t}{dt} \\
\boldsymbol{g_y} &\boldsymbol{= \alpha(g_K - g_A - g_L) + g_A}
\end{aligned}
$$

Finding steady state

For whatever model you're working with...

1. gather your **endogenous** factor accumulation equations, $\dot{Z}$
2. for each, divide through by Z to get an expression for $\boldsymbol{g_Z}$
3. look at the right-hand side: is there a **ratio** of exogenous vs. endogenous variables?

$$g_K = s_I \left(\frac{K_t}{A_t L_t}\right)^{\alpha - 1} - \delta, \qquad g_A = \theta s_R^{\lambda} \left(\frac{L_t^{\lambda}}{A_t^{1-\phi}}\right)$$

4. in steady state, g_Z must be **constant** $\longrightarrow$ growth rate of that ratio must be **zero**, so take logs and derivatives of the ratio and set equal to zero:

$$\frac{K_t}{A_t L_t} \rightarrow g_K - g_A - g_L = 0 \qquad \frac{L_t^{\lambda}}{A_t^{1-\phi}} \rightarrow \lambda g_L - (1-\phi) g_A = 0$$

5. **draw your phase diagram**: ratio on the x-axis, g_Z on the y-axis

Finding the BGP

6. intersection of two lines on phase diagram $\rightarrow \boldsymbol{g_Z^{\text{ss}}}$ (y-axis) and **ratio**$^{\text{ss}}$ (x-axis)
7. if necessary, use **ratio**$^{\text{ss}}$ to write Z_t^{BGP} in terms of params. & exog. variables
8. recall what we derived re: **GDP per capita** (maybe need to include h, $\boldsymbol{s_R}$, ...)

$$y_t = \tilde{k}_t^{\alpha} A_t, \qquad g_y = \alpha(g_K - g_A - g_L) + g_A$$

9. so, in steady state, $g_y^{\text{ss}} = g_A$ (potentially endogenous)...
10. ...and, along the BGP,

$$y_t^{\text{BGP}} = \left(\tilde{k}^{\text{ss}}\right)^{\alpha} A_t^{\text{BGP}}$$

$$\ln y_t^{\text{BGP}} = \alpha \ln \tilde{k}^{\text{ss}} + \ln A_0 + g_A^{\text{ss}}\, t$$

References

Weinberg, Steven. 2018. *Third thoughts*. Cambridge, MA: Harvard University Press.