**Economics 332 – Theory of Income III**
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**Midterm 2**

## Instructions

*This exam consists of two parts. You have 50 minutes to complete this exam and another 20 minutes to upload the exam to Canvas.*

## Ramsey Problem with No Capital (100 points)

Consider a deterministic economy where all agents have utility

$$\sum_{t=0}^{\infty} \beta^t U(c_t, n_t)$$

over consumption and labor sequences. Assume that $U$ is increasing in $c$, decreasing in $n$, and strictly concave in $(c, n)$. Feasibility is given by

$$c_t + g_t = A_t n_t$$

where $g_t$ stands for government purchases. Assume that the government taxes labor at rate $\{\tau_t\}$ to finance the stream of government purchases $\{g_t\}$.

1. Write down the Arrow-Debreu budget constraint of the household. Use $p_t$ for the AD price of consumption at $t$ in units of date-0 goods, and $w_t$ for time-$t$ before-tax wages in term of consumption at date $t$. Use $\tau_t$ for the proportional tax on labor income. [*Solution should be 1 equation. 10 points.*]

   *Solution*:
   $$\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t (1 - \tau_t) n_t w_t$$

2. Write down the first-order condition for $c_t$ and $n_t$ for the household problem. Write down the first-order condition for the firm problem. [*Solution should be 3 equations. 10 points.*]

   *Solution*:

   $$\begin{aligned} \beta^t U_c(c_t, n_t) &= \lambda p_t \\ -\beta^t U_n(c_t, n_t) &= \lambda p_t (1 - \tau_t) w_t \\ w_t &= A_t \end{aligned}$$

3. Derive the implementability condition for this economy. [*Solution should be 1 equation. 10 points.*]

   *Solution*:
   $$0 = \sum_{t=0}^{\infty} \beta^t \left( U_c(c_t, n_t) c_t + U_n(c_t, n_t) n_t \right)$$

4. Set up the Ramsey problem. [*Solution should be a maximization problem. 10 points.*]

*Solution*:

$$\begin{aligned}
\max_{(c_t,n_t)_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t U(c_t, n_t) \\
\text{s.t.} \quad & \sum_t \beta^t \left[c_t U_c(c_t, n_t) + n_t U_n(c_t, n_t)\right] = 0 \\
& c_t + g_t = A_t n_t \quad \forall t
\end{aligned}$$

5. What are the first order conditions with respect to $c_t$ and $n_t$? [*Solution should be 2 equations. 10 points.*]

   *Solution*:

$$\begin{aligned}
U_{ct} + \lambda\left[U_{ct} + c_t U_{cct} + n_t U_{cnt}\right] &= \mu_t \\
U_{nt} + \lambda\left[U_{nt} + c_t U_{cnt} + n_t U_{nnt}\right] &= -A_t \mu_t
\end{aligned}$$

6. For the rest of the problem, let utility take the form $U(c,n) = \log(c) - \alpha n$ where $\alpha > 0$. Solve for the Lagrange multiplier $\lambda$ as a function of the sequence $\{g_t\}$. [*Solution should be 1 equation. Hint: combine the implemntability condition, first order conditions, and feasibility constraint. 20 points.*]

   *Solution*: We can first simplify the implementability condition using the functional form of utility. We have $U_c c = 1$ and $U_n n = -\alpha n$. Implementability is therefore

$$0 = \sum_{t=0}^{\infty} \beta^t (1 - \alpha n_t)$$

   and the first-order conditions to the Ramsey problem are

$$\begin{aligned}
\frac{1}{c_t} &= \mu_t \\
-\alpha + \lambda[-\alpha] &= -A_t \mu_t
\end{aligned}$$

   which we can combine together to get

$$c_t = \frac{A_t}{\alpha(1+\lambda)}. \tag{1}$$

   Substitute the feasibility constraint into the implementability constraint, then substitute (1) in for $c_t$ and simplify:

$$\begin{aligned}
0 &= \sum_{t=0}^{\infty} \beta^t \left(1 - \alpha \frac{c_t + g_t}{A_t}\right) \\
&= \sum_{t=0}^{\infty} \beta^t \left(1 - \alpha \frac{\frac{A_t}{\alpha(1+\lambda)} + g_t}{A_t}\right) \\
&= \sum_{t=0}^{\infty} \beta^t \left(1 - \frac{1}{1+\lambda} - \alpha \frac{g_t}{A_t}\right).
\end{aligned}$$

Now solve for $\lambda$:

$$0 = \sum_{t=0}^{\infty} \beta^t \left(1 - \frac{1}{1+\lambda} - \alpha \frac{g_t}{A_t}\right)$$

$$\left(1 - \frac{1}{1+\lambda}\right) \frac{1}{1-\beta} = \alpha \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}$$

$$\frac{1}{1+\lambda} = 1 - \alpha(1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}$$

$$\lambda = \left(1 - \alpha(1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}\right)^{-1} - 1.$$

7. Compare the solution for $c_t$ in the Ramsey problem with the solution for $c_t$ in the first best. If there is a difference between the two, explain what causes it. [*Solution should be 2 equations and 1 sentence. 10 points.*]

   *Solution*: Solving for $c_t$, we have

$$c_t = \frac{A_t}{\alpha(\lambda+1)} = \frac{A_t}{\alpha} \left(1 - \alpha(1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}\right)$$

   The first best solution is

$$c_t = \frac{1}{\alpha}.$$

8. How do the level of taxes $\tau_t$, $\lambda$ and the level consumption $c_t$ and labor $n_t$ depend on the present value of $g$, where present value is taken with respect to beta? Are taxes constant or vary over time? [*Solution should be 4 equation and 2 sentences. 10 points.*]

   *Solution*: We have already derived two of these equations:

$$c_t = \frac{A_t}{\alpha(\lambda+1)} = \frac{A_t}{\alpha} \left(1 - \alpha(1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}\right)$$

$$\lambda = \left(1 - \alpha(1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}\right)^{-1} - 1$$

   Then labor is given by the market clearing condition:

$$n_t = \frac{c_t + g_t}{A_t}$$

   and taxes can be backed out from household first-order conditions:

$$\tau = 1 - \frac{\alpha c_t}{A_t} = \alpha(1-\beta) \sum_{t=0}^{\infty} \beta^t \frac{g_t}{A_t}$$

   where we can substitute in the equation for $c_t$ above.

9. Is the Ramsey problem, under the current assumptions, a convex maximization problem, i.e. whether the first order conditions are sufficient for a solution? Explain your reasoning. [*Solution should be 1 sentence. 10 points.*]

   *Solution*: Yes, since implementability is a linear constraint.

## Aggregation

Consider a pure endowment economy where preferences are indexed by a vector $\theta^i = \{\theta^i_\ell\}_{\ell=1}^m$ and a scalar $\sigma^i$ as follows:

$$u^i(x_1, x_2, \ldots, x_m) = \left[\sum_{\ell=1}^m \left(x^i_\ell - \theta^i_\ell\right)^{\frac{\sigma_i - 1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i - 1}}$$

1. Write down the social planner's problem. Use $\lambda_i$ to denote the weight on a household of type $i$ and $\bar{e}_\ell$ to denote the total endowment of commodity $\ell$. [*Solution should be a set of choice variables, an objective, and its constraint(s).*]

   *Solution*:

$$\max_{x^i \in X_i} \sum_i \lambda_i \left[\sum_{\ell=1}^m \left(x^i_\ell - \theta^i_\ell\right)^{\frac{\sigma_i - 1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i - 1}} \quad \text{s.t.} \quad \sum_i x^i_\ell \leq \bar{e}_\ell \;\; \forall \ell.$$

2. Write down the planner's FOC for $x^i_\ell$. Use $\gamma_\ell$ to denote the Lagrange multiplier on the feasibility constraint for commodity $\ell$. [*Solution should be 1 equation.*]

   *Solution*:

$$\lambda_i (x^i_\ell - \theta^i_\ell)^{-\frac{1}{\sigma_i}} \left[\sum_{\ell=1}^m \left(x^i_\ell - \theta^i_\ell\right)^{\frac{\sigma_i - 1}{\sigma_i}}\right]^{\frac{1}{\sigma_i - 1}} = \gamma_\ell$$

   or, equivalently,

$$\begin{aligned}(x^i_\ell - \theta^i_\ell) &= \lambda_i^{\sigma_i} \gamma_\ell^{-\sigma_i} \left[\sum_{\ell=1}^m \left(x^i_\ell - \theta^i_\ell\right)^{\frac{\sigma_i - 1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i - 1}} \\ &= \lambda_i^{\sigma_i} \gamma_\ell^{-\sigma_i} u^i. \end{aligned} \tag{2}$$

3. Define $\bar{\theta}_\ell \equiv \sum_i \theta^i_\ell$. Using the FOC for $x^i_\ell$ and the feasibility constraint for commodity $\ell$, can you write $\gamma_\ell$ as a function of $\bar{e}_\ell$ and $\bar{\theta}_\ell$? If so, do it. If not, what assumption on $\{\sigma_i\}$ would allow you to do so? **Use that assumption in all that follows.** [*Solution should be "yes" with 1 equation or "no" with 1 assumption and 1 equation.*]

   *Solution*: Sum (2) over $i$:

$$\begin{aligned}\sum_i u_i \lambda_i^{\sigma_i} \gamma_\ell^{-\sigma_i} &= \sum_i (x^i_\ell - \theta^i_\ell) \\ &= \bar{e}_\ell - \theta_\ell.\end{aligned}$$

   Note, though, that we can't factor out $\gamma_\ell^{-\sigma_i}$ unless $\sigma_i = \sigma$ for all $i$. Under this additional assumption,

$$\gamma_\ell^{-\sigma} \sum_i u_i \lambda_i^\sigma = \bar{e}_\ell - \theta_\ell$$

$$\gamma_\ell = (\bar{e}_\ell - \theta_\ell)^{-\frac{1}{\sigma}} \left(\sum_i u_i \lambda_i^\sigma\right)^{\frac{1}{\sigma}}.$$

4. Derive an expression for $\gamma_\ell/\gamma_k$. Does it depend on $\{\lambda_i\}$? [*Solution should be 1 equation.*]

   *Solution*: No, it does not depend on the weights:

$$\begin{aligned}\frac{\gamma_\ell}{\gamma_k} &= \left(\frac{\bar{e}_\ell - \theta_\ell}{\bar{e}_k - \theta_k}\right)^{-\frac{1}{\sigma}} \left(\frac{\sum_i u_i \lambda_i^\sigma}{\sum_i u_i \lambda_i^\sigma}\right)^{\frac{1}{\sigma}} \\ &= \left(\frac{\bar{e}_\ell - \theta_\ell}{\bar{e}_k - \theta_k}\right)^{-\frac{1}{\sigma}}.\end{aligned}$$

5. In the competitive equilibrium version of this model, do relative prices depend on the distribution of wealth? Do we have aggregation? [*Solution should be no more than 3 sentences.*]

   *Solution*: Recall that relative prices in the competitive equilibrium are such that $p_\ell/p_k = \gamma_\ell/\gamma_k$. Since $\gamma_\ell/\gamma_k$ is independent of $\{\lambda_i\}$, the relative prices are as well. We have aggregation.