TA Session: Ordinary Differential Equations

Econ 30400: Mathematical Methods for Economics

Levi Crews (Chicago)

September 2020

A bit about me

- Rising fourth-year PhD in the economics department
- Focused on macro, trade, and quantitative spatial models
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A bit about this class

Math camp at many other programs: a mandatory class that you **must pass**; ensures that you have the **minimum** required mathematical knowledge to **survive** the first-year curriculum

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Math camp here: an **optional**, highly-concentrated preview of **all** the mathematical tools you could need to **ace** the first-year curriculum

- you don't need to know everything we cover!
- most of what you will need will be covered again in your core classes
- don't get overwhelmed, don't get distracted

What we'll cover in TA sessions

- **Sept. 14 (Monday)**: ODEs following Barro & Sala-i-Martin (2004, A.1)
- **Sept. 15 (Tuesday)**: continue with ODEs
- **Sept. 16 (Wednesday)**: computational dynamic programming
- **Sept. 17 (Thursday)**: continue with computational DP

Let’s dive in!

Definitions

- **differential equation**: relates an unknown function to its derivatives
  - **ordinary**: only one unknown function

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$$

  - **partial**: more than one unknown function

$$\rho v(a, z, t) = \max_{c} u(c) + [z + ra - c]\partial_a v(a, z, t) + \mu(z)\partial_z v(a, z, t) + \frac{1}{2}\sigma^2(z)\partial_{zz} v(a, z, t) + \partial_t v(a, z, t)$$

- **order**: of an ODE, the order of its highest derivative
  - *N.b.* can always rewrite an nth-order ODE as system of first-order ODE

A simple taxonomy of ODEs

Consider the ODE

$$a_1(t) \cdot \dot{y}(t) + a_2(t) \cdot y(t) + x(t) = 0. \tag{1}$$

This is a **linear, first-order ODE**. If ...

- $a_j(t) = a_j$ for $j = 1, 2$, then (1) has **constant coefficients**;
- further, $x(t) = a_3$, then (1) is **autonomous**;
- even further, $x(t) = 0$, then (1) is **homogeneous**.

An example of a **nonlinear, first-order ODE** is

$$\ln [\dot{y}(t)] + \frac{1}{y(t)} = 0.$$

Objective: Find the behavior of $y(t)$.

Solution methods:

- **Graphical.** Draw proto-phase diagram, which works for both linear and nonlinear ODES, but *only autonomous* ones.
- **Analytical.** Find exact formula for $y(t)$, but only for a *limited set of functional forms* (including linear) $\implies$ can approximate nonlinear by linearizing.
- **Numerical.** Try ODE solvers from Matlab, Julia, and/or Python packages.

Graphical: proto-phase diagrams

Consider an autonomous ODE

$$\dot{y}(t) = f[y(t)]$$

where f may or may not be linear.
Now **plot f as a function of y.**

- **locally unstable**: $\frac{\partial \dot{y}}{\partial y}|_{y^*} > 0$
- **locally stable**: $\frac{\partial \dot{y}}{\partial y}|_{y^*} < 0$
- **Solow-Swan model: one of each!**

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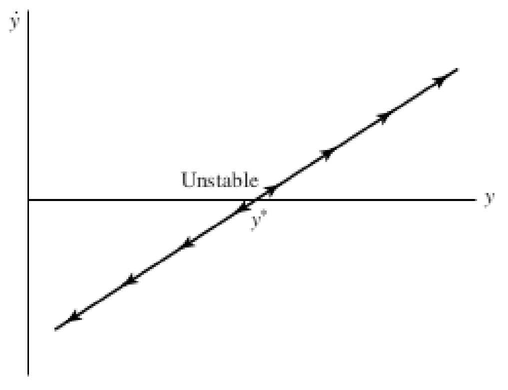

Arrows show direction that y moves over time

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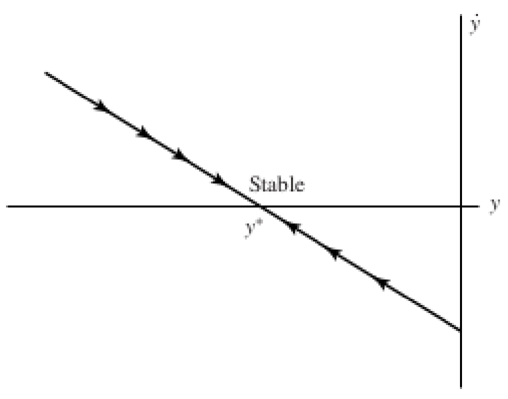

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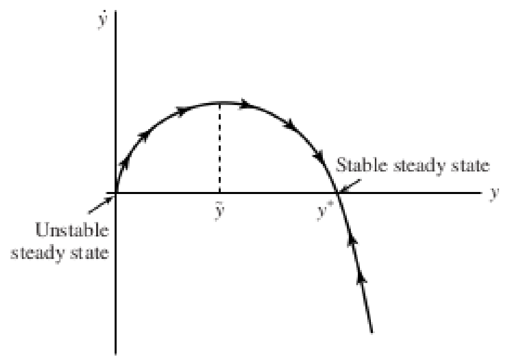

Arrows show direction that y moves over time

Analytical: Linear, first-order ODE with constant coefficients

$$0 = \dot{y}(t) + a \cdot y(t) + x(t)$$

Steps:

1. Separate variables
2. Multiply by *integrating factor* e^{at} and integrate
3. By FTC, we know RHS; by definition of $x(t)$ we can compute LHS, call it $X(t)$
4. Solve for $y(t)$

Analytical: Linear, first-order ODE with constant coefficients

$$0 = \dot{y}(t) + a \cdot y(t) + x(t)$$

Steps:

1. Separate variables

$$-x(t) = \dot{y}(t) + a \cdot y(t)$$

2. Multiply by *integrating factor* e^{at} and integrate
3. By FTC, we know RHS; by definition of $x(t)$ we can compute LHS, call it $X(t)$
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2. Multiply by *integrating factor* e^{at} and integrate

$$-\int e^{at} \cdot x(t)\, dt = \int e^{at} \cdot [\dot{y}(t) + a \cdot y(t)]\, dt$$

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$$X(t) + b_1 = e^{at} y(t) + b_0$$

4. Solve for $y(t)$

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4. Solve for $y(t)$

$$y(t) = -e^{-at}X(t) + be^{-at}$$

Analytical: Linear, first-order ODE with variable coefficients

$$0 = \dot{y}(t) + a(t) \cdot y(t) + x(t)$$

Steps:

1. Separate variables
2. Multiply by *integrating factor* $e^{\int_0^t a(\tau)\, d\tau}$ and integrate
3. By FTC, we know RHS; by definition of $x(t)$ we can compute LHS, call it $\tilde{X}(t)$
4. Solve for $y(t)$

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Steps:

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2. Multiply by *integrating factor* $e^{\int_0^t a(\tau)\,d\tau}$ and integrate

$$-\int e^{\int_0^t a(\tau)\,d\tau} \cdot x(t)\,dt = \int e^{\int_0^t a(\tau)\,d\tau} \cdot [\dot{y}(t) + a(t) \cdot y(t)]\,dt$$

3. By FTC, we know RHS; by definition of $x(t)$ we can compute LHS, call it $\tilde{X}(t)$
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$$\tilde{X}(t) + b_1 = e^{\int_0^t a(\tau)\, d\tau} y(t) + b_0$$

4. Solve for $y(t)$

Analytical: Linear, first-order ODE with variable coefficients

$$0 = \dot{y}(t) + a(t) \cdot y(t) + x(t)$$

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4. Solve for $y(t)$

$$y(t) = -e^{-\int_0^t a(\tau)\,d\tau}\tilde{X}(t) + be^{-\int_0^t a(\tau)\,d\tau}$$

Boundary value problems

- The solutions we derived in the last two cases are **general solutions**.
- Specify the arbitrary constant $b \implies$ get a **particular** (or **exact**) **solution**.
- How to pick b: specify a **boundary value** of $y(t)$
  - **initial condition**: for initial value y_0,

$$y_0 \equiv y(0) = X(0) + b \implies b = y_0 - X(0)$$

  - **terminal condition**: for terminal value y_T at date T,

$$y_T \equiv y(T) = -e^{-aT} X(T) + be^{-aT} \implies b = X(T) + e^{aT} y_T$$

Solving systems of linear ODEs

Now we'll study a system of linear, first-order ODEs of the form

$$\dot{y}_1(t) = a_{11}y_1(t) + \ldots + a_{1n}y_n(t) + x_1(t)$$
$$\vdots$$
$$\dot{y}_n(t) = a_{n1}y_1(t) + \ldots + a_{nn}y_n(t) + x_n(t)$$

or, in matrix notation,

$$\dot{y}(t) = A \cdot y(t) + x(t) \tag{2}$$

Again there are three types of solution procedures:

- **Graphical.** Draw phase diagrams, works for linear and nonlinear, but only for 2×2 systems of autonomous equations
- **Analytical.** Generally only for linear systems
- **Numerical.** Shooting algorithms and time-elimination methods

Phase diagrams: A diagonal example

Use the diagonal, autonomous system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Steps to draw phase diagram in (y_1, y_2)-space:

1. Draw $\dot{y}_1 = 0$ nullcline
2. Draw arrows in each of the two regions split by the nullcline
3. Repeat for $\dot{y}_2 = 0$ nullcline
4. Join the two pictures
5. Use BV to identify exact solution

Phase diagrams: A diagonal example

Use the diagonal, autonomous system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} >0 & 0 \\ 0 & >0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Steps to draw phase diagram in (y_1, y_2)-space:

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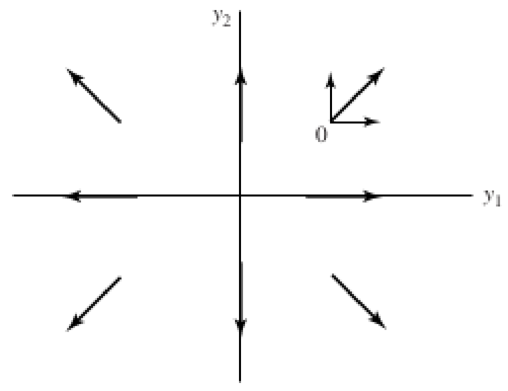

Unstable

Phase diagrams: A diagonal example

Use the diagonal, autonomous system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} <0 & 0 \\ 0 & <0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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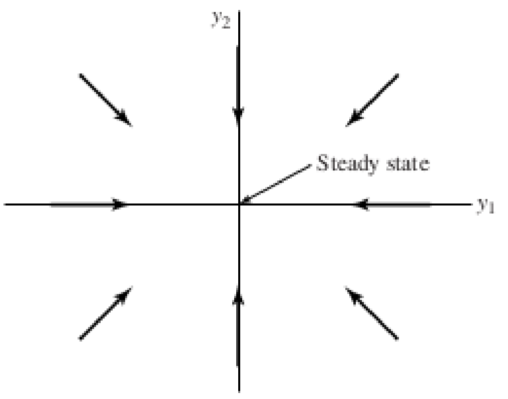

Stable

Phase diagrams: A diagonal example

Use the diagonal, autonomous system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} < 0 & 0 \\ 0 & > 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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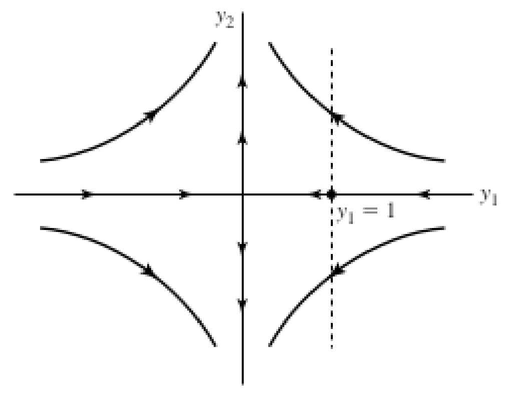


Saddle-path stable

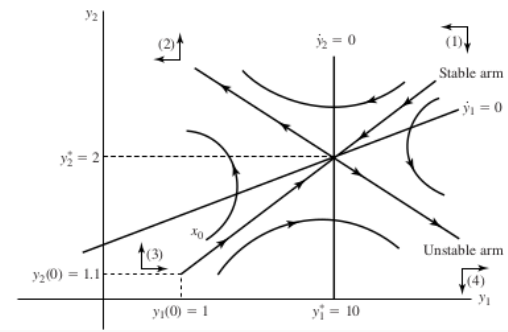
Phase diagrams: A nondiagonal example

Use the nondiagonal system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0.06 & -1 \\ -0.004 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1.4 \\ 0.04 \end{bmatrix}$$

with boundary conditions

- $y_1(0) = 1$, and
- $\lim_{t\to\infty}[e^{-0.06t} \cdot y_1(t)] = 0$.



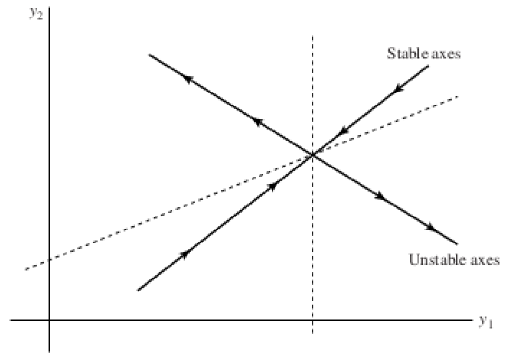
Phase diagrams: A nondiagonal example

Use the nondiagonal system

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0.06 & -1 \\ -0.004 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1.4 \\ 0.04 \end{bmatrix}$$

with boundary conditions . . .

Note. Leaving just the stable and unstable arms looks like a distorted version of the saddle-path stable figure from the diagonal case above. **Why might that be?**



Phase diagrams: A nonlinear example (Neoclassical growth model)

Use the nonlinear system

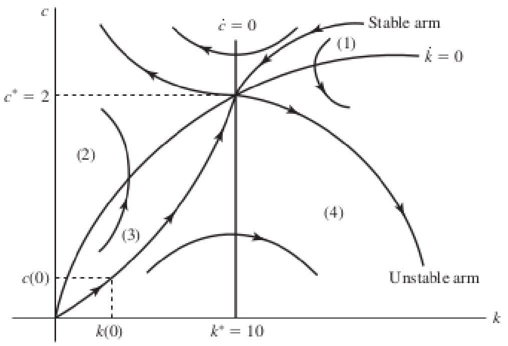
$$\dot{k}(t) = k(t)^{0.3} - c(t)$$
$$\dot{c}(t) = c(t) \cdot [0.3k(t)^{-0.7} - 0.06]$$

with boundary conditions $k(0) = 1$ and

$$\lim_{t \to \infty} [e^{-0.06t} \cdot k(t)] = 0.$$

The steps to draw the phase diagram are **the same as before**.

- $\dot{k} = 0$ nullcline: $c = k^{0.3}$
- $\dot{c} = 0$ nullcline: $k = 10$



Analytical solutions of linear, homogeneous systems

Use the linear, homogeneous system

$$\dot{y}(t) = A \cdot y(t).$$

Assume that A is **diagonalizable**: it can be written as

$$A = V \Lambda V^{-1}$$

where

- V is the matrix of eigenvectors of A
- Λ is the diagonal matrix of eigenvalues of A

1. Find the eigenvalues of the matrix A; call them $\lambda_1, \ldots, \lambda_n$.
2. Find the corresponding eigenvectors; use them to construct V.
3. Rewrite the system using the change of variables $z(t) = V^{-1} \cdot y(t)$:

$$\dot{z}(t) = \Lambda \cdot z(t)$$

4. Solution is $z_i(t) = b_i \cdot e^{\lambda_i t}$ for $i = 1, \ldots, n$; gather into matrix as $z(t) = Eb$.
5. Get general solution: $y = VEb$

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Analytical solutions of linear, nonhomogeneous systems

Use the linear, nonhomogeneous system

$$\dot{y}(t) = A \cdot y(t) + x(t)$$

Assume as before that A is diagonalizable:

$$A = V \Lambda V^{-1}$$

1. Find the eigenvalues and eigenvectors of A
2. Rewrite the system using the change of variables $z(t) = V^{-1} \cdot y(t)$:

$$\dot{z}(t) = \Lambda \cdot z(t) + V^{-1} \cdot x(t)$$

3. Solution for $i = 1, \ldots, n$ is

$$z_i(t) = e^{\lambda_i t} \int e^{-\lambda_i \tau} [V_i^{-1} \cdot x(\tau)] \, d\tau + e^{\lambda_i t} b_i$$

   which we can gather into $z(t) = E\hat{X} + Eb$

4. Get general solution: $y = VE\hat{X} + VEb$

Analytical solutions of linear, nonhomogeneous systems

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Analytical solutions of linear, nonhomogeneous systems

Work through example on p.590–592 of Barro & Sala-i-Martin (2004)

Connecting the graphical and analytical solutions

When we diagonalize A, we do a change of basis $\implies$ "shift" the axes

- new axes are the eigenvectors of A
- the elements of the new diagonal matrix that governs the system are the eigenvalues of A

Stability properties depend on signs of eigenvalues.

If the two eigenvalues are ...

1. **real** and **positive** $\implies$ unstable
2. **real** and **negative** $\implies$ stable
3. **real** with **opposite signs** $\implies$ saddle-path stable
4. **complex** with **negative real parts** $\implies$ oscillating convergence
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6. **complex** and **zero real parts** $\implies$ ellipses around S.S.
7. **equal** $\implies$ $y_i(t) = (b_{i1} + b_{i2}t)e^{\lambda t}$

Connecting the graphical and analytical solutions

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Oh, so your system is nonlinear?

Consider the system $\dot{y}_i(t) = f_i[y_1(t), \ldots, y_n(t)]$ for $i = 1, \ldots, n$, where functions f_i are nonlinear. **Now linearize around the steady state.**

$$\begin{aligned}
\dot{y}_1(t) &= f_1^* + (f_1^*)_{y_1}(y_1 - y_1^*) + \ldots + (f_1^*)_{y_n}(y_n - y_n^*) + R_1 \\
&\vdots \\
\dot{y}_n(t) &= f_n^* + (f_n^*)_{y_1}(y_1 - y_1^*) + \ldots + (f_n^*)_{y_n}(y_n - y_n^*) + R_n
\end{aligned}$$

with

- $y_i^* \equiv$ steady-state value of y_i
- $(f_i^*)_{y_j} \equiv$ partial derivative of f_i w.r.t. y_j at s.s.
- $f_i^* \equiv$ steady-state value of f_i
- $R_i \equiv$ Taylor residuals

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\end{aligned}$$

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- $y_i^* \equiv$ steady-state value of y_i
- $(f_i^*)_{y_j} \equiv$ partial derivative of f_i w.r.t. y_j at s.s.
- $f_i^* \equiv$ steady-state value of f_i, **which equals zero**
- $R_i \equiv$ Taylor residual, **which is approximately zero around s.s.**

Oh, so your system is nonlinear?

Consider the system $\dot{y}_i(t) = f_i[y_1(t), \ldots, y_n(t)]$ for $i = 1, \ldots, n$, where functions f_i are nonlinear. **Now linearize around the steady state.**

$$
\begin{aligned}
\dot{y}_1(t) &= (f_1^*)_{y_1}(y_1 - y_1^*) + \ldots + (f_1^*)_{y_n}(y_n - y_n^*) \\
&\vdots \\
\dot{y}_n(t) &= (f_n^*)_{y_1}(y_1 - y_1^*) + \ldots + (f_n^*)_{y_n}(y_n - y_n^*)
\end{aligned}
$$

with

- $y_i^* \equiv$ steady-state value of y_i
- $(f_i^*)_{y_j} \equiv$ partial derivative of f_i w.r.t. y_j at s.s.

Oh, so your system is nonlinear? Now it's (approximately) linear!

Consider the system $\dot{y}_i(t) = f_i[y_1(t), \ldots, y_n(t)]$ for $i = 1, \ldots, n$, where functions f_i are nonlinear. **Now linearize around the steady state.**

$$\dot{y}(t) = A \cdot (y - y^*), \qquad A \equiv \begin{bmatrix} (f_1^*)_{y_1} & \cdots & (f_1^*)_{y_n} \\ \vdots & \ddots & \vdots \\ (f_n^*)_{y_1} & \cdots & (f_n^*)_{y_n} \end{bmatrix}$$

with

- $y_i^* \equiv$ steady-state value of y_i
- $(f_i^*)_{y_j} \equiv$ partial derivative of f_i w.r.t. y_j at s.s.

Our ur-example: The neoclassical growth model

Use the nonlinear system

$$\dot{k}(t) = k(t)^{0.3} - c(t)$$
$$\dot{c}(t) = c(t) \cdot [0.3k(t)^{-0.7} - 0.06]$$

with boundary conditions $k(0) = 1$ and

$$\lim_{t\to\infty} [e^{-0.06t} \cdot k(t)] = 0.$$

Steady state: $(k^*, c^*) = (10, 2)$

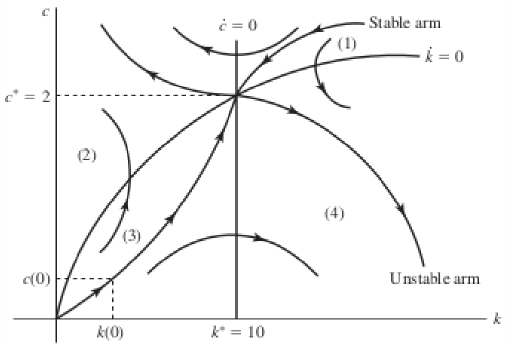
$$\begin{aligned}\dot{k}(t) &\approx 0.3(k^*)^{-0.7}(k - k^*) - (c - c^*) \\ &= 0.06k - c + 1.4 \\ \dot{c}(t) &\approx c^* \left[0.3 \cdot (-0.7)(k^*)^{-1.7}\right](k - k^*) \\ &\quad - 0(c - c^*) \\ &= -0.008k + 0.08\end{aligned}$$

All together . . .

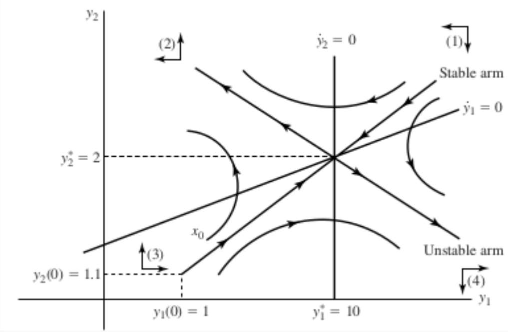
$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0.06 & -1 \\ -0.008 & 0 \end{bmatrix} \begin{bmatrix} k \\ c \end{bmatrix} + \begin{bmatrix} 1.4 \\ 0.08 \end{bmatrix}$$

Wait . . . we've seen that system before!

Slide 15: NGM



Slide 14: That system



Numerics: Time-elimination method

Read p.593–596 of Barro & Sala-i-Martin (2004)