The dynamics of inequality

Gabaix, Lasry, Lions, & Moll (*Econometrica* 2016)

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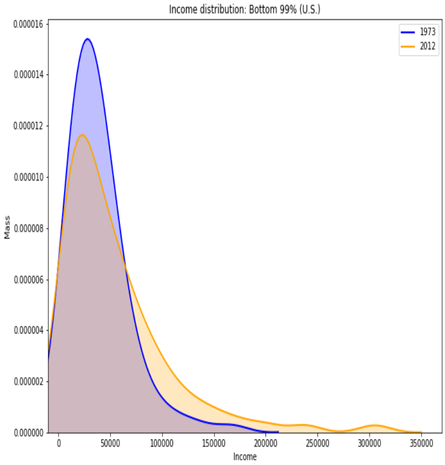
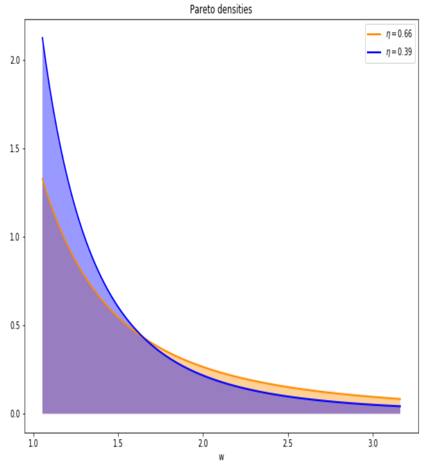
The rapid rise in top income inequality



- U.S. pre-tax national income (source: World Inequality Database)
- 1973 (blue) vs. 2012 (orange)
- **takeaway: mass has shifted to right tail**
- hard to see: right tail (top decile) is roughly Pareto

Background: What is a Pareto tail?



- Pareto tail above pth percentile:

$$\mathbb{P}\{w_{it} > w\} = Cw^{-1/\eta}$$

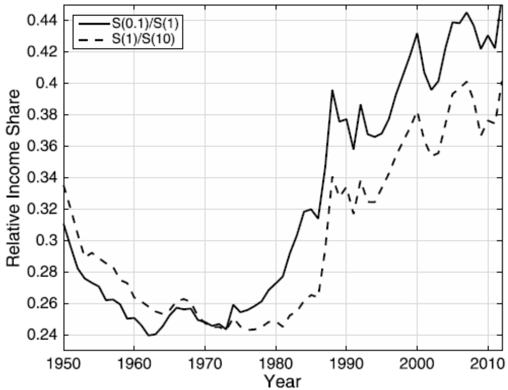
- **higher $\eta \implies$ fatter tail**
- equivalent characterization:

$$S\left(\frac{p}{100}\right) = 10^{\eta-1} \cdot S\left(\frac{p}{10}\right) = 10^{2(\eta-1)} \cdot S(p)$$

where

  - $S(p)$ = income share of pth percentile

The rapid rise in top income inequality



- relative income shares of top decile (1950–2012 U.S.)
- claim: the top decile of the U.S. income distribution is (roughly) Pareto
- exact Pareto: these lines would **overlap**
- rise in η $\implies$ **rise in "top inequality"**

What explains the rapid rise in top income inequality (η)?

- Try tool off-shelf: **standard random growth model**
  - known to generate Pareto tail
- **main result**: converges too slowly!
  - ergodic theory & PDEs $\Longrightarrow$ analytic formula for speed of convergence
  - Laplace transform $\Longrightarrow$ tail converges *even slower* than average
- extension: add superstars and convex skill prices
  - can converge arbitrarily quickly (or slowly)

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Standard random growth model: Components

- continuum of workers i
- log wage $x_{it} = \log w_{it}$
- **random growth**:

$$dx_{it} = \mu\, dt + \sigma\, dZ_{it}$$

- **stabilizing forces**:
    - death rate δ (newborn $\sim \psi(x)$)
    - reflecting barrier $\underline{x} = 0$
- can allow arbitrary $\mu(x,t)$, $\sigma(x,t)$, jumps, reinjections

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- **stationary tail:** Gabaix (2009)

$$\mathbb{P}(w_{it} > w) \sim C w^{-1/\eta}$$

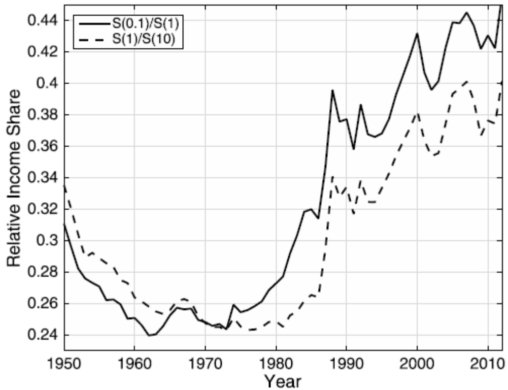
- **top inequality:**

$$\eta = \frac{\sigma^2}{-\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}$$

- If dist. of w has Pareto tail above pth percentile:

$$\frac{S(p/10)}{S(p)} = 10^{\eta - 1}$$

Standard random growth model: Stationary distribution



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Thought experiment

- **into time $t = 0$:** economy in steady state with dist. p_0 and params. μ_0, σ_0^2, etc.
- **at time $t = 0$:** increase $\sigma_\infty^2 > \sigma_0^2$
- **as time $t \to \infty$:** converge to new stationary distribution p_∞
- **question:** how quickly $p_0 \to p_\infty$?
- measure of distance: the L^1-norm

$$\|p(t) - p_\infty\| := \int_{-\infty}^{\infty} |p(x,t) - p_\infty(x)|\, dx$$

	1973	2012	Source
η	0.39	0.66	$1 + \log_{10} S(0.1)/S(1)$
σ	0.1	0.158	HPV (2010)
μ	0002	0.002	match η
δ	1/30	1/30	match retirement

Kolmogorov forward equation

- **Kolmogorov forward equation:**

$$\underbrace{p_t}_{\text{evolution of } p} = - \underbrace{\mu \cdot p_x}_{\text{outflow from drift}} + \underbrace{\frac{\sigma^2}{2} \cdot p_{xx}}_{\text{Ito term}} - \underbrace{\delta p}_{\text{outflow from death}} + \underbrace{\delta \psi}_{\text{inflow from birth}}$$

with initial condition $p(x,0) = p_0(x)$ and boundary condition

$$0 = -\mu \cdot p(0,t) + \frac{\sigma^2}{2} \cdot p_x(0,t), \quad \forall t$$

- Rewrite using **differential operator:**

$$p_t = \mathcal{A}^* p + \delta \psi, \qquad \mathcal{A}^* := \mu \cdot p_x + \frac{\sigma^2}{2} \cdot p_{xx} - \delta p$$

- $\mathcal{A}^*$ is linear $\implies$ **"transition matrix" for continuous-state process**

Main result: Average speed of convergence

The cross-sectional dist. $p(x,t)$ *converges to its stationary dist.* $p_\infty(x)$ *in the* L^1*-norm for any initial dist.* $p_0(x)$*. The rate of convergence*

$$\lambda := -\lim_{t\to\infty} \frac{1}{t} \log \| p(x,t) - p_\infty(x) \|$$

without a reflecting barrier is

$$\lambda = \delta$$

and with a reflecting barrier ($\underline{x} = 0$) *is*

$$\lambda = \frac{\mu^2}{2\sigma^2} \cdot \mathbf{1}_{\{\mu<0\}} + \delta.$$

- **special case:** reflecting barrier, $\mu < 0$, $\delta = 0$
  - top inequality:

$$\eta = -\sigma^2/(2\mu)$$

  - speed of convergence:

$$\lambda = \frac{\sigma^2}{8\eta^2}$$

  - **High inequality goes hand in hand with slow transitions!**
  - calibrated half-life of L^1-distance:

$$t_{1/2} = \frac{\log 2}{\lambda} \approx 96 \text{ years}$$

Intuition: Convergence of finite-state Markov chains

- **recall**: $\mathcal{A}^*$ is a linear operator $\implies$ like an infinite-dimensional matrix
- chop up the state space: $x_{it} \in \{x_1, \ldots, x_n\}$
- now use $n \times n$ transition matrix $\boldsymbol{A}$: KFE is approximately

$$p_t = \boldsymbol{A}' p + \delta\psi, \qquad p(0) = p_0$$

- assume $\boldsymbol{A}$ is **symmetric & diagonalizable** $\implies$

$$p(t) = \sum_{j=1}^{n} c_j e^{\Lambda_j t} v_j, \qquad p_0 = \sum_{j=1}^{n} c_j v_j \quad \text{with} \quad \begin{array}{l} \text{eigenvalues } \Lambda_j \in \mathbb{R} \\ \text{eigenvectors } v_j \end{array}$$

- **stationary distribution:** $\Lambda_1 = 0$, $p_\infty = c_1 v_1$
- **speed:** with eigenvalues ordered $|\Lambda_j| < |\Lambda_k|$ if $j < k$ and $p_0 \not\perp v_2$,

$$\text{for large } t: \quad p(t) \approx p_\infty + c_2 e^{\Lambda_2 t} v_2 \implies \boxed{\lambda = -\Lambda_2}$$

Tail convergence: Even slower than average

- Maybe tail converges faster than average?
- **Laplace transform:** without reflecting barrier

$$\hat{p}(\xi, t) := \int_{-\infty}^{\infty} e^{-\xi x} p(x, t)\, dx$$
$$= \mathbb{E}\left[w_{it}^{-\xi}\right]$$
$$= -\xi\text{th moment of income distribution } (\xi < 0)$$

- more negative $\xi \implies$ more weight on p's tail

- *The rate of convergence of the $-\xi$th moment of the distribution of income is*

$$\lambda(\xi) := \xi\mu - \xi^2 \frac{\sigma^2}{2} + \delta$$
$$< \delta$$
$$= \text{ avg. rate of convergence}$$

- **Tail converges even slower than average!**

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