Solutions: Problem Set 3

Economics 27000

Levi Crews

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Question 1

1. "There always exists a set of transfers that makes everyone weakly better off from free trade."

   True. It's true that some agents may be made worse off by free trade in the specific-factors model, but there are still (weakly) positive gains from trade in the aggregate. Accordingly, we can make lump-sum transfers from the winners to the losers in such a way that everyone is (weakly) better off.

2. "A rise in the relative price of food will benefit workers in the food sector but hurt workers in the cloth sector."

   False. Labor is the flexible factor in the specific-factors model, so wages (and workers' welfare) must equalize across sectors. So it can never be that a change causes some workers to gain and others to lose in this model.

3. "An increase in the stock of capital leads to a decrease in the rental rate of both capital and land."

   True. It's easy to see for capital, since the supply of capital has increased. But notice, too, that MPL in cloth production will increase, so wages must rise and labor will shift into cloth production. Accordingly, labor shifts out of food production. With higher wages but fixed commodity prices, it follows that demand for land will fall, driving down its rental rate. One can also see this by totally differentiating the zero-profit condition of the food producer.

Question 2

For each of the following production functions, check if it satisfies

- constant returns to scale
- (strictly) diminishing returns to each factor.

Note that $Z > 0$ is a measure of total factor productivity in each of the production functions below.

Solution. In what follows, assume $\lambda > 1$.

1. **Linear**:
$$F(K, L) = Z[\alpha K + (1 - \alpha)L], \quad \alpha \in [0, 1].$$

   - **constant returns to scale**: Yes.
$$\begin{aligned} F(\lambda K, \lambda L) &= Z[\alpha(\lambda K) + (1 - \alpha)(\lambda L)] \\ &= \lambda Z[\alpha K + (1 - \alpha)L] \\ &= \lambda F(K, L) \end{aligned}$$

   - **strictly diminishing marginal products**: No.
$$\begin{aligned} \frac{\partial^2 F}{\partial K^2} &= \frac{\partial}{\partial K}[Z\alpha] = 0 \\ \frac{\partial^2 F}{\partial L^2} &= \frac{\partial}{\partial L}[Z(1 - \alpha)] = 0 \end{aligned}$$

2. **Leontief**:
$$F(K, L) = Z \min\{K, L\}.$$

   - **constant returns to scale**: Yes.
$$\begin{aligned} F(\lambda K, \lambda L) &= Z \min\{\lambda K, \lambda L\} \\ &= \lambda Z \min\{K, L\} \\ &= \lambda F(K, L) \end{aligned}$$

   - **strictly diminishing marginal products**: No. Let's start by deriving the marginal product of labor for fixed capital K^* (the case of capital is analogous). If $L < K^*$, an extra unit of labor yields an extra Z units of output, so $\text{MP}_L = Z$. If $L \geq K^*$, an extra unit of labor yields no extra output, so $\text{MP}_L = 0$. In either case, the marginal product is constant, not strictly diminishing. Note, though, that MP_L is decreasing overall: it's a step function with a downward jump from Z to 0 at $L = K^*$.

3. **Cobb-Douglas**:
$$F(K, L) = ZK^\alpha L^{1-\alpha}, \quad \alpha \in [0, 1].$$

   - **constant returns to scale**: Yes.
$$\begin{aligned} F(\lambda K, \lambda L) &= Z[\alpha(\lambda K) + (1 - \alpha)(\lambda L)] \\ &= \lambda Z[\alpha K + (1 - \alpha)L] \\ &= \lambda F(K, L) \end{aligned}$$

   - **strictly diminishing marginal products**: Yes.
$$\begin{aligned} \frac{\partial^2 F}{\partial K^2} &= \frac{\partial}{\partial K}\left[\alpha Z(L/K)^{1-\alpha}\right] = -\alpha(1 - \alpha)ZL^{1-\alpha}K^{\alpha-2} < 0 \\ \frac{\partial^2 F}{\partial L^2} &= \frac{\partial}{\partial L}\left[(1 - \alpha)Z(K/L)^\alpha\right] = -\alpha(1 - \alpha)ZK^\alpha L^{-\alpha-1} < 0 \end{aligned}$$

4. **Constant Elasticity of Substitution (CES)**:

$$F(K,L) = Z\left[\alpha K^{\sigma} + (1-\alpha)L^{\sigma}\right]^{1/\sigma}, \quad \sigma \in [-\infty, 1], \;\; \alpha \in [0,1].$$

- **constant returns to scale**: Yes.

$$\begin{aligned} F(\lambda K, \lambda L) &= Z[\alpha(\lambda K)^{\sigma} + (1-\alpha)(\lambda L)^{\sigma}]^{1/\sigma} \\ &= Z[(\lambda)^{\sigma}(\alpha K^{\sigma} + (1-\alpha)L^{\sigma})]^{1/\sigma} \\ &= \lambda Z[\alpha K^{\sigma} + (1-\alpha)L^{\sigma}]^{1/\sigma} \\ &= \lambda F(K,L) \end{aligned}$$

- **strictly diminishing marginal products**: Yes.

$$\begin{aligned} \frac{\partial^2 F}{\partial K^2} &= \frac{\partial}{\partial K}\left[\alpha Z K^{\sigma-1}\left(\alpha K^{\sigma} + (1-\alpha)L^{\sigma}\right)^{\frac{1-\sigma}{\sigma}}\right] \\ &= -\alpha(1-\alpha)(1-\sigma)ZK^{\sigma-2}L^{\sigma}\left[\alpha K^{\sigma} + (1-\alpha)L^{\sigma}\right]^{\frac{1}{\sigma}-2} \\ &< 0 \\ \frac{\partial^2 F}{\partial L^2} &= \frac{\partial}{\partial L}\left[(1-\alpha)ZL^{\sigma-1}\left(\alpha K^{\sigma} + (1-\alpha)L^{\sigma}\right)^{\frac{1-\sigma}{\sigma}}\right] \\ &= -\alpha(1-\alpha)(1-\sigma)ZL^{\sigma-2}K^{\sigma}\left[\alpha K^{\sigma} + (1-\alpha)L^{\sigma}\right]^{\frac{1}{\sigma}-2} \\ &< 0 \end{aligned}$$

Note: The first three production functions are all limiting cases of CES. The linear production function is CES with $\sigma = 1$. The Leontief production function is CES when $\sigma \to -\infty$. The Cobb-Douglas production function is CES when $\sigma \to 0$.

5. **Stone-Geary**:

$$F(K,L) = Z(K-\bar{K})^{\alpha}(L-\bar{L})^{1-\alpha}, \quad \bar{K}, \bar{L} > 0, \;\; \alpha \in [0,1].$$

- **constant returns to scale**: No, because $\bar{K}, \bar{L} > 0$.

$$\begin{aligned} F(\lambda K, \lambda L) &= Z[\alpha(\lambda K - \bar{K}) + (1-\alpha)(\lambda L - \bar{L})] \\ &\neq Z[\alpha(\lambda K - \lambda\bar{K}) + (1-\alpha)(\lambda L - \lambda\bar{L})] \\ &= \lambda Z[\alpha(K - \bar{K}) + (1-\alpha)(L - \bar{L})] \\ &= \lambda F(K,L) \end{aligned}$$

- **strictly diminishing marginal products**: Yes, if $K > \bar{K}$ and $L > \bar{L}$.

$$\frac{\partial^2 F}{\partial K^2} = \frac{\partial}{\partial K}\left[\alpha Z\left(\frac{L-\bar{L}}{K-\bar{K}}\right)^{1-\alpha}\right] = -\alpha(1-\alpha)Z(L-\bar{L})^{1-\alpha}(K-\bar{K})^{\alpha-2} < 0$$

$$\frac{\partial^2 F}{\partial L^2} = \frac{\partial}{\partial L}\left[(1-\alpha)Z\left(\frac{K-\bar{K}}{L-\bar{L}}\right)^{\alpha}\right] = -\alpha(1-\alpha)Z(K-\bar{K})^{\alpha}(L-\bar{L})^{-\alpha-1} < 0$$

Question 3

Consider the specific-factors model (2 goods, 3 factors, where labor is the flexible factor) for a given country with the following Cobb-Douglas production technologies:

$$Q_C = Z_C K^{1/4} L_C^{3/4} \qquad \text{(cloth production)}$$
$$Q_F = Z_F T^{1/4} L_F^{3/4} \qquad \text{(food production)}$$

where Z_C and Z_F are the productivities in both sectors, $L = L_C + L_F$ is the aggregate endowment of labor, K is the aggregate endowment of capital, and T is the aggregate endowment of natural resources.

1. Derive the equation of the (inverse) relative supply curve for this economy (p_C/p_F as a function of Q_C/Q_F). What can you say about the shape of this curve?

   Solution. Equate the marginal productivity of labor in both sectors:

   $$p_C \frac{3}{4} Z_C \left(\frac{K}{L_C}\right)^{1/4} = p_F \frac{3}{4} Z_F \left(\frac{T}{L_F}\right)^{1/4}.$$

   Solve for L_C/L_F:

   $$\frac{L_C}{L_F} = \left(\frac{p_C Z_C}{p_F Z_F}\right)^4 \frac{K}{T}.$$

   Construct the ratio of Q_C/Q_F from the production functions:

   $$\frac{Q_C}{Q_F} = \frac{Z_C}{Z_F}\left(\frac{K}{T}\right)^{1/4}\left(\frac{L_C}{L_F}\right)^{3/4}.$$

   Substitute for L_C/L_F:

   $$\frac{Q_C}{Q_F} = \left(\frac{p_C}{p_F}\right)^3 \left(\frac{Z_C}{Z_F}\right)^4 \left(\frac{K}{T}\right).$$

   Solve for p_C/p_F:

   $$\frac{p_C}{p_F} = \left(\frac{Q_C}{Q_F}\right)^{1/3} \left(\frac{Z_C}{Z_F}\right)^{-4/3} \left(\frac{K}{T}\right)^{-1/3}.$$

   So the relative supply curve is increasing and concave in $(Q_C/Q_F, p_C/p_F)$-space.

2. Suppose that a country opens up to international trade. Assume that this country is small relative to its trading partners, so that the prices of cloth and food are now fixed at world prices: $p_C = 2$ and $p_F = 1$. Further assume that all consumers in this country have Leontief preferences over food and cloth: $U(Q_C, Q_F) = \min\{Q_C, 4Q_F\}$. How will the country's pattern of trade depend on the parameters of the model (K, L, T, Z_C, Z_F)? How does the size of the labor force L affect the pattern of trade?

   Solution. With $p_C^* = 2$ and $p_F^* = 1$, we have

   $$8 = \left(\frac{Q_C}{Q_F}\right)\left(\frac{Z_C}{Z_F}\right)^{-4}\left(\frac{K}{T}\right)^{-1}.$$

   With the given Leontief preferences, $Q_C = 4Q_F$. So ...

- if $2 > \left(\frac{Z_C}{Z_F}\right)^{-4}\left(\frac{K}{T}\right)^{-1}$, then the country will export cloth and import food;
- if $2 = \left(\frac{Z_C}{Z_F}\right)^{-4}\left(\frac{K}{T}\right)^{-1}$, then we cannot pin down the pattern of trade;
- if $2 < \left(\frac{Z_C}{Z_F}\right)^{-4}\left(\frac{K}{T}\right)^{-1}$, then the country will export food and import cloth.

Note that the pattern of trade is independent of the aggregate endowment of labor, L.

3. How will an increase in productivity in the cloth sector (Z_C) affect the welfare of all three factor owners? Contrast your response with the welfare change induced by an increase in the aggregate capital stock K. Assume that world prices remain fixed throughout.

Solution. We consider workers, land owners, and capital owners in turn.

- **Workers.** It will be sufficient to check how wages respond. To get an expression for wages, set the wage equal to the marginal revenue product in each sector and use the labor market clearing condition $L = L_C + L_F$:

$$
\begin{aligned}
L_C &= \left(\frac{3}{4}\frac{p_C^* Z_C}{w}\right)^4 K \\
L_F &= \left(\frac{3}{4}\frac{p_F^* Z_F}{w}\right)^4 T \\
&\implies \\
w &= \frac{3}{4}\left[\frac{(2Z_C)^4 K + (Z_F)^4 T}{L}\right]^{1/4}.
\end{aligned}
$$

Now just check how the wage responds to increases in Z_C and K:

$$
\begin{aligned}
\frac{\partial w}{\partial Z_C} &= 12\frac{K}{L}Z_C^3\left[\frac{(2Z_C)^4 K + (Z_F)^4 T}{L}\right]^{-3/4} \\
&= \frac{81}{16}\frac{K}{L}Z_C^3 w^{-3} \\
&> 0, \\
\frac{\partial w}{\partial K} &= 3\frac{1}{L}Z_C^4\left[\frac{(2Z_C)^4 K + (Z_F)^4 T}{L}\right]^{-3/4} \\
&> 0.
\end{aligned}
$$

So workers are better off in both cases.

- **Land owners.** We check how the rental rate of land, r_T, responds. Land is paid its marginal revenue product:

$$
r_T = \frac{1}{4}p_F^* Z_F \left(\frac{L_F}{T}\right)^{3/4}.
$$

Substitute for L_F:

$$
r_T = \frac{27}{256}Z_F^4 w^{-3}.
$$

Take derivatives, using what we know about wages from above:

$$\frac{\partial r_T}{\partial Z_C} = -\frac{81}{256} Z_F^4 w^{-4} \frac{\partial w}{\partial Z_C} < 0$$
$$\frac{\partial r_T}{\partial K} = -\frac{81}{256} Z_F^4 w^{-4} \frac{\partial w}{\partial K} < 0.$$

So land owners are worse off in both cases.

- **Capital owners.** We check the how the rental rate of capital, r_K, responds. Capital is paid its marginal revenue product:

$$r_K = \frac{1}{4} p_C^* Z_C \left(\frac{L_C}{K}\right)^{3/4}.$$

Substitute for L_C:

$$r_K = \frac{27}{128} Z_C^4 w^{-3}.$$

Take derivatives, using what we know about wages from above:

$$\begin{aligned}\frac{\partial r_K}{\partial Z_C} &= \frac{27}{64} Z_C^3 w^{-3} - \frac{81}{128} Z_C^4 w^{-4} \frac{\partial w}{\partial Z_C} \\ &= \frac{27}{64} Z_C^3 w^{-3} \left(1 - \frac{3}{2} \frac{Z_C}{w} \frac{\partial w}{\partial Z_C}\right) \\ &= \frac{27}{64} Z_C^3 w^{-3} \left(1 - \frac{243}{32} \frac{K}{L} Z_C^4 w^{-4}\right),\end{aligned}$$

the sign of which depends on the sign of the term in parentheses, and

$$\frac{\partial r_K}{\partial K} = -\frac{81}{128} Z_C^4 w^{-4} \frac{\partial w}{\partial K} < 0.$$